Ellen P. Eliot

Class V

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ALGEBRA

FOR

SECONDARY SCHOOLS

BY

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PREFACE

The present work is intended to meet the needs of High Schools and Academies of the highest grade. While in the main similar to the author's "Essentials of Algebra," many additional topics have been introduced, and improvements made; attention is especially invited to the following:

1. The product by inspection of two binomials of the form $mx + n$ and $px + q$ (§ 100).

2. In the chapter on Factoring will be found the factoring of expressions of the forms $x^4 + ax^2y^2 + y^4$ and $ax^2 + bx + c$, when the factors do not involve surds (§§ 115, 117).

These forms are considered later in §§ 298 and 300.

The solution of equations by factoring is also taken up in this chapter.

In § 107 will be found many new varieties of examples.

3. In § 176 will be found a set of problems in which the solutions are negative, fractional, or zero.

4. In the chapter on Evolution will be found the square root by inspection of polynomials of the form

$$a^2 + b^2 + c^2 + 2ab + 2ac + 2bc,$$

and the cube root by inspection of polynomials of the form $a^3 + 3a^2b + 3ab^2 + b^3$ (§§ 212, 223).

The development of the rules for the square and cube root of polynomials and arithmetical numbers leaves nothing to be

desired from a theoretical point of view. (See §§ 213, 214, 217, 224, 225, 228.)

5. In the solution of quadratic equations by formula (§ 289), the equation is in the form $ax^2 + bx + c = 0$.

6. In all the theoretical work in Chapter XXI, the quadratic equation is in the form $ax^2 + bx + c = 0$.

7. In the chapter on Ratio and Proportion, in several of the demonstrations of theorems, fractions are used in place of ratio symbols.

8. In §§ 386 and 387 will be found the same proof of the Binomial Theorem for Positive Integral Exponents as is given in the "Essentials of Algebra"; those wishing a more complete proof, in which the general law of coefficients is proved for any two consecutive terms, will find it in § 447.

9. The proof of the Theorem of Undetermined Coefficients given in § 396 is the same as that given in the "Essentials of Algebra"; a more rigorous proof is given in § 450.

10. The author has thought it best to omit the proof of the Binomial Theorem for Fractional and Negative Exponents, as a rigorous demonstration is beyond the capacity of pupils in preparatory schools.

11. In Chapter XXXII will be found Highest Common Factor and Lowest Common Multiple by Division; and also the reduction of a fraction to its lowest terms, when the numerator and denominator cannot be readily factored by inspection.

Any teacher who so desires can take this work in connection with Chapters IX and X.

Chapter XXXIII also contains the proof of (1), § 235, for all values of m and n (§ 445); and the reduction of a fraction whose denominator is irrational to an equivalent fraction

having a rational denominator, when the denominator is the sum of a rational expression and a surd of the nth degree, or of two surds of the nth degree (§ 446).

An important feature of the work is the prominence given to graphical methods; in Chapter XIII will be found the graph of a linear equation with two unknown numbers, and also of a linear expression involving one unknown number.

In §§ 184, 185, and 186 will be found the graphical representations of the solutions of simultaneous linear equations, including inconsistent and indeterminate equations.

The subject is taken up for quadratic equations in §§ 303 to 305, and 314 to 316.

To meet the demands of many schools, a number of physical problems have been introduced; these will be found at the end of Exercises 62, 128, 129, and 145.

At the end of the chapter on Variation will be found a set of problems in physics in which the principles of variation are employed; and also several illustrations of the application of graphs in physics. All the above work in physics has been prepared by Professor Robert A. Millikan, of the University of Chicago.

In nearly every set of numerical equations, beginning with Exercise 58, will be found examples in which other letters than x, y, and z are used to represent unknown numbers.

The examples and problems are about 4000 in number; and no example is a duplicate of any in the author's "Academic Algebra" or "Essentials of Algebra."

There is throughout the work a much greater variety of examples than in the above treatises.

An important and useful feature of the work is the Index,

which contains references to all operations and important definitions.

Metric tables for use in connection with the Physical Problems of the text, will be found on page 458.

The author desires to express his thanks to the many teachers in secondary schools, whose suggestions in the preparation of the work have been of the greatest service.

WEBSTER WELLS.

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ALGEBRA FOR SECONDARY SCHOOLS

ALGEBRA

I. DEFINITIONS AND NOTATION

1. In **Algebra,** the operations of Arithmetic are abridged and generalized by means of **Symbols.**

SYMBOLS REPRESENTING NUMBERS

2. The symbols usually employed to represent numbers are the *Arabic Numerals* and the *letters of the Alphabet.*

The numerals represent known or determinate numbers.

The letters represent numbers which may have any values whatever, or numbers whose values are to be found.

SYMBOLS REPRESENTING OPERATIONS

3. The **Sign of Addition,** $+$, is read "*plus.*"

Thus, $a+b$ signifies that the number represented by b is to be added to the number represented by a; $a+b+c$ signifies that the number represented by b is to be added to the number represented by a, and then the number represented by c added to the result; and so on.

The result of addition is called the **Sum.**

We shall use the expression "*the number a,*" or simply "a," to signify "*the number represented by a,*" etc.

4. The **Sign of Subtraction,** $-$, is read "*minus.*"

Thus, $a-b$ signifies that the number b is to be subtracted from the number a; $a-b-c$ signifies that b is to be subtracted from a, and then c subtracted from the result; and so on.

5. The **Sign of Multiplication,** $\times$, is read "*times,*" or "*multiplied by.*"

Thus, $a \times b$ signifies that the number a is to be multiplied by the number b; $a \times b \times c$ signifies that a is to be multiplied by b, and the result multiplied by c; and so on.

The result of multiplication is called the **Product.**

The sign of multiplication is usually omitted in Algebra, except between two numbers expressed in Arabic numerals.

Thus, $2x$ signifies 2 multiplied by x; but the product of 2 by 3 could not be expressed 2 3.

A point is often used for the sign $\times$; thus $1 \cdot 2$ signifies 1×2.

6. The **Sign of Division,** $\div$, is read "*divided by.*"

Thus, $a \div b$ signifies that the number a is to be divided by the number b; it is also read "*the quotient of* a *by* b."

The division of a by b is also expressed $\frac{a}{b}$.

EQUATIONS

7. The **Sign of Equality,** $=$, is read "*equals.*"

Thus, $a = b$ signifies that the number a equals the number b.

8. An **Equation** is a statement that two numbers are equal.

The *first member* of an equation is the number to the left of the sign of equality, and the *second member* is the number to the right of that sign; thus, in the equation $2x - 3 = 5$, the first member is $2x - 3$, and the second member 5.

AXIOMS

9. An **Axiom** is a truth which is assumed as self-evident.

Algebraic operations are based on the following axioms:

1. *Any number equals itself.*
2. *Any number equals the sum of all its parts.*
3. *Any number is greater than any of its parts.*
4. *Two numbers which are equal to the same number, or to equal numbers, are equal.*

5. *If the same number, or equal numbers, be added to equal numbers, the resulting numbers will be equal.*

6. *If the same number, or equal numbers, be subtracted from equal numbers, the resulting numbers will be equal.*

7. *If equal numbers be multiplied by the same number, or equal numbers, the resulting numbers will be equal.*

8. *If equal numbers be divided by the same number, or equal numbers, the resulting numbers will be equal.*

SOLUTION OF PROBLEMS BY ALGEBRAIC METHODS

10. The following examples will illustrate the use of Algebraic symbols in the solution of problems.

The utility of the process consists in the fact that the unknown numbers are represented by *symbols,* and that the various operations are stated in *Algebraic language.*

1. The sum of two numbers is 30, and the greater exceeds the less by 4; what are the numbers?

We will represent the less number by x.

Then the greater will be represented by $x+4$.

By the conditions of the problem, the sum of the *less* number and the *greater* is 30; this is stated in Algebraic language as follows:

$$x+x+4=30. \qquad (1)$$

Now, $x+x=x\times 2$; for to multiply an arithmetical number by 2, we add it twice.

Again, $x\times 2=2\times x$, or $2x$ (§ 5); for the product of two arithmetical numbers is the same in whichever order they are multiplied.

Therefore, $x+x=2x$; and equation (1) can be written

$$2x+4=30.$$

The members of this equation, $2x+4$ and 30, are equal numbers; if from each of them we subtract the number 4, the resulting numbers will be equal (Ax. 6, § 9).

Therefore, $\qquad 2x=30-4$, or $2x=26$.

Dividing the equal numbers $2x$ and 26 by 2 (Ax. 8, § 9), we have

$$x=13.$$

Hence, the less number is 13, and the greater is $13+4$, or 17.

The written work will stand as follows:

Let $x =$ the less number.

Then, $x + 4 =$ the greater number.

By the conditions, $x + x + 4 = 30$, or $2x + 4 = 30$.

Whence, $2x = 26$.

Dividing by 2, $x = 13$, the less number.

Whence, $x + 4 = 17$, the greater number.

2. The sum of the ages of A and B is 109 years, and A is 13 years younger than B; find their ages.

Let x represent the number of years in B's age.

Then, $x - 13$ will represent the number of years in A's age.

By the conditions of the problem, the sum of the ages of A and B is 109 years.

Whence, $x - 13 + x = 109$, or $2x - 13 = 109$.

Adding 13 to both members (Ax. 5, § 9),

$$2x = 122.$$

Dividing by 2, $x = 61$, the number of years in B's age.

And, $x - 13 = 48$, the number of years in A's age.

The written work will stand as follows:

Let $x =$ the number of years in B's age.

Then, $x - 13 =$ the number of years in A's age.

By the conditions, $x - 13 + x = 109$, or $2x - 13 = 109$.

Whence, $2x = 122$.

Dividing by 2, $x = 61$, the number of years in B's age.

Therefore, $x - 13 = 48$, the number of years in A's age.

It must be carefully borne in mind that x can only represent an *abstract number.*

Thus, in Ex. 2, we do not say "let x represent B's *age*," but "let x represent the *number of years* in B's age."

3. A, B, and C together have \$66. A has one-half as much as B, and C has 3 times as much as A. How much has each?

Let $x =$ the number of dollars A has.

Then, $2x =$ the number of dollars B has,

and $3x =$ the number of dollars C has.

By the conditions, $x + 2x + 3x = 66.$

But the sum of x, twice x, and 3 times x is 6 times x, or $6x$.

Whence, $6x = 66.$

Dividing by 6, $x = 11$, the number of dollars A has.

Whence, $2x = 22$, the number of dollars B has,

and $3x = 33$, the number of dollars C has.

(By letting x represent the number of dollars A has, in Ex. 3, we avoid fractions.)

EXERCISE I

1. The greater of two numbers is 8 times the less, and exceeds it by 49; find the numbers.

2. The sum of the ages of A and B is 119 years, and A is 17 years older than B; find their ages.

3. Divide \$74 between A and B so that A may receive \$48 more than B.

4. Divide \$108 between A and B so that A may receive 5 times as much as B.

5. Divide 91 into two parts such that the smaller shall be one-sixth of the greater.

6. A man travels 112 miles by train and steamer; he goes by train 54 miles farther than by steamer. How many miles does he travel in each way?

7. The sum of three numbers is 69; the first is 14 greater than the second, and 28 greater than the third. Find the numbers.

8. The sum of the ages of A, B, and C is 134 years; B is 13 years younger than A, and 7 years younger than C. Find their ages.

9. A cow and sheep together cost $91, and the sheep cost one-twelfth as much as the cow; how much did each cost?

10. Divide $6.75 between A and B so that A may receive one-fourth as much as B.

11. A man has $2. After losing a certain sum, he finds that he has left 20 cents more than 3 times the sum which he lost. How much did he lose?

12. A, B, and C have together $140; A has 4 times as much as B, and C has as much as A and B together. How much has each?

13. A, B, and C have together $100; A has $10 less than C, and C has $25 more than B. How much has each?

14. At an election two candidates, A and B, had together 653 votes, and A was beaten by 395 votes. How many did each receive?

15. A field is 7 times as long as it is wide, and the distance around it is 240 feet. Find its dimensions.

16. My horse, carriage, and harness are worth together $325. The horse is worth 6 times as much as the harness, and the carriage is worth $65 more than the horse. How much is each worth?

17. The sum of three numbers is 87; the third number is one-eighth of the first, and the second number 15 less than the first. Find the numbers.

18. At a certain election, three candidates, A, B, and C, received together 436 votes; A had a majority over B of 14 votes, and over C of 3 votes. How many did each receive?

19. The sum of the ages of A, B, and C is 110 years; B's age exceeds twice C's by 12 years, and A is 9 years younger than B. Find their ages.

20. A pole 77 feet long is painted red, white, and black; the red is one-fifth of the white, and the black 21 feet more than the red. How many feet are there of each color?

21. Divide 70 into three parts such that the third part shall be one-fifth of the first, and one-fourth of the second.

22. Divide $ 7.55 between A, B, and C so that C may receive one-half as much as A, and B $ 2.95 less than A and C together.

23. A, B, and C have together $ 22.50; B has $ 1.50 more than A, and C has $ 8 less than twice the amount that A has How much has each?

24. The profits of a shopkeeper in a certain year were one-third as great as in the preceding year, and $ 515 less than in the following year. If the total profits for the three years were $ 2615, what were the profits in each year?

25. The sum of four numbers is 96. The first is 4 times the fourth, and exceeds the third by 20; and the second exceeds the sum of the first and fourth by 4. Find the numbers.

26. Divide $ 468 between A, B, C, and D, so that A may receive one-fifth as much as B, B one-fifth as much as C, and C one-fifth as much as D.

DEFINITIONS

11. The continued product of a number by itself any number of times is called a **Power** of the number.

An **Exponent** is a number written at the right of, and above another number, to indicate what power of the latter is to be taken; thus,

a^2, read "*a square*," or "*a second power*," denotes $a \times a$;
a^3, read "*a cube*," or "*a third power*," denotes $a \times a \times a$;
a^4, read "*a fourth*," "*a fourth power*," or "*a exponent* 4," denotes $a \times a \times a \times a$, etc.

If no exponent is expressed, the *first* power is understood. Thus, a is the same as a^1.

12. Symbols of Aggregation.

The *parentheses* ( ), the *brackets* [ ], the *braces* { }, and the *vinculum* ——, indicate that the numbers enclosed by them are to be taken collectively; thus,

$$(a+b)\times c,\ [a+b]\times c,\ \{a+b\}\times c, \text{ and } \overline{a+b}\times c$$

all indicate that the result obtained by adding b to a is to be multiplied by c.

EXERCISE 2

What operations are signified by the following?

1. $2x^5y^6$.
2. $m(x-y)$.
3. $\frac{ab}{cd}$.
4. $x+(y-z)$.
5. $x-(y+z)$.
6. $(m-n)^8$.
7. $\frac{a}{b}-\frac{c}{d}$.
8. $\left(\frac{x+y}{x-y}\right)^2$.
9. $(2a+3b)(4c-5d)$
10. $\left(\frac{1}{x}+\frac{1}{y}\right)z^4$.

Write the following in symbols:

11. The result of subtracting 6 times n from 5 times m.

12. Three times the product of the eighth power of m and the ninth power of n.

13. The quotient of the sum of a and b, divided by the sum of c and d.

14. The product of $3x+y$ and z^2.

15. The result of subtracting $y-z$ from x.

16. The product of $a-b$ and $c-d$.

17. The result of adding the quotient of m by n, and the quotient of x by y.

18. The square of $m+n$.

19. The cube of $a-b+c$.

20. The fourth power of the quotient of a divided by x.

21. The product of the quotient of 1 by x and the quotient of 1 by y.

ALGEBRAIC EXPRESSIONS

13. **An Algebraic Expression,** or simply an **Expression,** is a number expressed in algebraic symbols; as,

$$2,\ a,\ \text{or}\ 2x^2 - 3ab + 5.$$

14. The **Numerical Value** of an expression is the result obtained by substituting particular numerical values for the letters involved in it, and performing the operations indicated.

1. Find the numerical value of the expression

$$4a + \frac{6c}{b} - d^3,$$

when $a = 4$, $b = 3$, $c = 5$, and $d = 2$.

We have, $4a + \frac{6c}{b} - d^3 = 4 \times 4 + \frac{6 \times 5}{3} - 2^3 = 16 + 10 - 8 = 18.$

If the expression involves *parentheses*, the operations indicated *within* the parentheses must be performed first.

2. Find the numerical value, when $a = 9$, $b = 7$, and $c = 4$, of

$$(a - b)(b + c) - \frac{a + b}{b - c}.$$

We have, $a - b = 2$, $b + c = 11$, $a + b = 16$, and $b - c = 3$.
Then the numerical value of the expression is

$$2 \times 11 - \frac{16}{3} = 22 - \frac{16}{3} = \frac{50}{3}.$$

EXERCISE 3

Find the numerical values of the following when $a = 6$, $b = 3$, $c = 4$, $d = 5$, $m = 3$, and $n = 2$:

1. $a^2b - cd^2$. **2.** $2abcd$. **3.** $3ab + 4bc - 5cd$.

4. $a^m b^n$.

5. $a^b + b^c$.

6. $\frac{a}{bc} + \frac{c}{ad}$.

7. $\frac{1}{b} + \frac{1}{c} - \frac{1}{d}$.

8. $\frac{15\,c^m}{28\,d^n}$.

9. $\frac{a}{b} - \frac{b}{c} + \frac{c}{d}$.

10. $\frac{1}{a} + \frac{1}{b} - \frac{1}{c} - \frac{1}{d}$.

11. $\frac{5\,c^2 d}{a^3} - \frac{cd^2}{2\,b^3}$.

12. $\frac{b^2}{a^2} + \frac{c^2}{b^2} - \frac{d^2}{c^2}$.

Find the numerical values of the following when $a = 5$, $b = 2$, $c = 3$, and $d = 4$:

13. $\left(\frac{2\,a + d}{2\,b + c}\right)^2$.

14. $(a^2 - b^2 - d^2)^3$.

15. $5\,a^2(a - b) - 2\,b^3(c + d)$.

16. $8\,(a - b)^2 + 3(c + d)^2$.

17. $(a - b)^2 + (2\,a - 3\,b)^2 - (b + c)^2$.

18. $(2\,a - b - c + d)\,(2\,a + b + c - d)$.

19. $\frac{8\,a + 3\,b - 6\,c}{9\,a - 4\,b - 3\,c}$.

20. $\frac{a - b}{a + b} + \frac{a - c}{a + c} + \frac{a - d}{a + d}$.

Find the numerical values of the following when $a = \frac{3}{4}$, $b = \frac{5}{2}$, $c = \frac{1}{3}$, and $x = 4$:

21. $\frac{a + c}{a - c} - \frac{a - c}{a + c}$.

22. $\frac{8\,a + 6\,b - 15\,c}{16\,a + 10\,b + 9\,c}$.

23. $x^3 + (2\,a + 3\,b)\,x^2 - (5\,a - 4\,c)\,x + \frac{8}{3}\,abc$.

24. $x^3 - \left(\frac{2}{a} + \frac{3}{b}\right)x^2 + \left(\frac{6}{a} - \frac{5}{b} + \frac{4}{c}\right)x - \frac{13}{abc}$.

II. POSITIVE AND NEGATIVE NUMBERS

15. There are certain concrete magnitudes which are capable of existing in two opposite states.

Thus, in financial transactions, we may have *assets* or *liabilities,* and *gains* or *losses;* we may have motion along a straight line in a certain direction, or in the *opposite* direction; etc.

In each of these cases, the effect of combining with a magnitude of a certain kind another of the opposite kind, is to diminish the former, destroy it, or reverse its state.

Thus, if to a certain amount of asset we add a certain amount of liability, the asset is diminished, destroyed, or changed into liability.

16. The signs + and −, besides denoting addition and subtraction, are also used, in Algebra, to distinguish between the opposite states of magnitudes like those of § 15.

Thus, we may indicate *assets* by the sign +, and *liabilities* by the sign −; for example, the statement that a man's assets are − \$100, means that he has liabilities to the amount of \$100.

EXERCISE 4

1. If a man has assets of \$400, and liabilities of \$600, how much is he worth?

2. If gains be taken as positive, and losses as negative, what does a gain of − \$100 mean?

3. In what position is a man who is − 3 miles north of a certain place?

4. In what position is a man who is − 50 feet west of a certain point?

5. How many miles north of a certain place is a man who goes 5 miles north, and then 9 miles south?

6. How many miles east of a certain place is a man who goes 11 miles west, and then 6 miles east?

17. Positive and Negative Numbers.

If the positive and negative states of any concrete magnitude be expressed *without reference to the unit,* the results are called *positive* and *negative numbers,* respectively.

Thus, in $+\$5$ and $-\$3$, $+5$ is a positive number, and -3 is a negative number.

For this reason the sign $+$ is called the *positive* sign, and the sign $-$ the *negative* sign.

If no sign is expressed, the number is understood to be positive; thus, 5 is the same as $+5$.

The negative sign must never be omitted before a negative number.

18. The **Absolute Value** of a number is the number taken independently of the sign affecting it.

Thus, the absolute value of -3 is 3.

ADDITION OF POSITIVE AND NEGATIVE NUMBERS

19. We shall give to addition in Algebra its arithmetical meaning, *so long as the numbers to be added are positive integers or positive fractions.*

We may then attach any meaning we please to addition involving other forms of numbers, provided the new meanings are not inconsistent with principles previously established.

20. In adding a positive number and a negative, or two negative numbers, our methods must be in accordance with the principles of § 15.

If a man has assets of \$5, and then incurs liabilities of \$3, he will be worth \$2.

If he has assets of \$3, and then incurs liabilities of \$5, he will be in debt to the amount of \$2.

If he has liabilities of \$5, and then incurs liabilities of \$3, he will be in debt to the amount of \$8.

Now with the notation of § 15, incurring liabilities of \$3 may be regarded as adding $-\$3$ to his property.

Whence, the sum of $+\$5$ and $-\$3$ is $+\$2$;

the sum of $-\$5$ and $+\$3$ is $-\$2$;

and the sum of $-\$5$ and $-\$3$ is $-\$8$.

Or, omitting reference to the *unit*,

$$(+5)+(-3)=+2;$$
$$(-5)+(+3)=-2;$$
$$(-5)+(-3)=-8.$$

To indicate the addition of $+5$ and -3, they must be enclosed in *parentheses* (§ 12).

We then have the following rules:

To add a positive and a negative number, subtract the less absolute value (§ 18) *from the greater, and prefix to the result the sign of the number having the greater absolute value.*

To add two negative numbers, add their absolute values, and prefix a negative sign to the result.

21. Examples.

1. Find the sum of $+10$ and -3.

Subtracting 3 from 10, the result is 7.

Whence, $(+10)+(-3)=+7.$

2. Find the sum of -12 and $+6$.

Subtracting 6 from 12, the result is 6.

Whence, $(-12)+(+6)=-6.$

3. Add -9 and -5.

The sum of 9 and 5 is 14.

Whence, $(-9)+(-5)=-14$

EXERCISE 5

Find the values of the following:

1. $(-6)+(-7)$.
2. $(+8)+(-3)$.
3. $(-9)+(+5)$.
4. $(+4)+(-11)$.
5. $(-13)+(-18)$.
6. $(-42)+(+57)$.
7. $(-34)+(+82)$.
8. $\left(-\frac{5}{9}\right)+\left(-\frac{1}{6}\right)$.
9. $\left(+\frac{9}{8}\right)+\left(-\frac{8}{7}\right)$.
10. $(-15\frac{1}{8})+(+12\frac{7}{9})$.
11. $(+17\frac{7}{8})+(-10\frac{5}{12})$.
12. $(-14\frac{5}{6})+(-21\frac{3}{14})$.

MULTIPLICATION OF POSITIVE AND NEGATIVE NUMBERS

22. If two algebraic expressions are multiplied together, the first is called the **Multiplicand**, and the second the **Multiplier.**

23. We shall retain for multiplication, in Algebra, its arithmetical meaning, *so long as the multiplier is a positive integer or a positive fraction.*

That is, to multiply a number by a positive integer is to *add* the multiplicand as many times as there are units in the multiplier.

For example, to multiply -4 by 3, we add -4 three times.

Thus, $(-4)\times(+3)=(-4)+(-4)+(-4)=-12$.

24. In Arithmetic, the product of two numbers is the same in whichever *order* they are multiplied.

Thus, 3×4 and 4×3 are each equal to 12.

If we could assume this law to hold for the product of a positive number by a negative, we should have

$$(+3)\times(-4)=(-4)\times(+3)=-12 \text{ (§ 23)} =-(3\times 4).$$

Then, if the above law is to hold, we must give the following meaning to multiplication by a negative number:

To multiply a number by a negative number is to multiply it by the absolute value (§ 18) *of the multiplier, and change the sign of the result.*

Thus, to multiply $+4$ by -3, we multiply $+4$ by $+3$, giving $+12$, and change the sign of the result.

That is, $$(+4) \times (-3) = -12.$$

Again, to multiply -4 by -3, we multiply -4 by $+3$, giving -12 (§ 23), and change the sign of the result.

That is, $$(-4) \times (-3) = +12.$$

25. From §§ 23 and 24 we derive the following rule:

To multiply one number by another, multiply together their absolute values.

Make the product **plus** *when the multiplicand and multiplier are of* **like** *sign, and* **minus** *when they are of* **unlike** *sign.*

26. Examples.

1. Multiply $+8$ by -5.

By the rule, $$(+8) \times (-5) = -(8 \times 5) = -40.$$

2. Multiply -7 by -9.

By the rule, $$(-7) \times (-9) = +(7 \times 9) = +63.$$

3. Find the numerical value when $a = 4$ and $b = -7$, of

$$(a+b)^3 - (a-b)^3.$$

We have, $$\begin{aligned}(a+b)^3-(a-b)^3 &= (4-7)(4-7)(4-7)-(4+7)(4+7)(4+7)\\ &= (-3)(-3)(-3) - 11 \times 11 \times 11\\ &= -27 - 1331 = -1358.\end{aligned}$$

EXERCISE 6

Find the values of the following:

1. $(+5) \times (-4)$.

2. $(-11) \times (+3)$.

3. $(-8)\times(-7)$.

4. $(+9)\times(-6)$.

5. $(-12)\times(+9)$.

6. $(-24)\times(-5)$.

7. $(-14)\times(+15)$.

8. $(+27)\times(-19)$.

9. $\left(-\frac{7}{8}\right)\times\left(-\frac{9}{5}\right)$.

10. $\left(+\frac{10}{11}\right)\times\left(-\frac{13}{15}\right)$.

11. $\left(-\frac{33}{35}\right)\times\left(-\frac{21}{44}\right)$.

12. $(-7\frac{4}{5})\times(-\frac{11}{26})$.

13. $(-6\frac{2}{9})\times(+6\frac{3}{7})$.

14. $(+1\frac{3}{14})\times(-1\frac{7}{18})$.

Find the numerical value when $a=2$, $b=-4$, $c=5$, and $d=-3$ of

15. b^a.

16. d^c.

17. $(bd)^2$.

18. $(abc)^3$.

19. $(2a-5b)(4c+3d)$.

20. $(a-b)(b-c)(c-d)$.

21. $(a+b)(c+d)-(a-c)(b-d)$.

22. $a^2b-3b^2c-2c^2d$.

23. $(a-c)^3+(b+d)^3$.

24. $3a^2b^2-5b^2c^2+4c^2d^2$.

III. ADDITION AND SUBTRACTION OF ALGEBRAIC EXPRESSIONS. PARENTHESES

27. A **Monomial,** or **Term,** is an expression (§ 13) whose parts are not separated by the signs + or −; as $2x^2$, $-3ab$, or 5.

$2x^2$, $-3ab$, and $+5$ are called the *terms* of the expression $2x^2-3ab+5$.

A **Positive Term** is one preceded by a + sign; as $+5a$.

If no sign is expressed, the term is understood to be positive.

A **Negative Term** is one preceded by a − sign; as $-3ab$.

The − sign must never be omitted before a negative term.

28. If two or more numbers are multiplied together, each of them, or the product of any number of them, is called a **Factor** of the product.

Thus, a, b, c, ab, ac, and bc are factors of the product abc.

29. Any factor of a product is called the **Coefficient** of the product of the remaining factors.

Thus, in $2ab$, 2 is the coefficient of ab, $2a$ of b, a of $2b$, etc.

30. If one factor of a product is expressed in *Arabic numerals,* and the other in *letters,* the former is called the *numerical coefficient* of the latter.

Thus, in $2ab$, 2 is the numerical coefficient of ab.

If no numerical coefficient is expressed, the coefficient 1 is understood; thus, a is the same as $1a$.

31. By § 25, $(-3)\times a=-(3\times a)=-3a$.

That is, $-3a$ is the product of -3 and a.

Then, -3 is the *numerical coefficient* of a in $-3a$.

Thus, *in a negative term as in a positive, the numerical coefficient includes the sign.*

32. **Similar** or **Like Terms** are those which either do not differ at all, or differ only in their numerical coefficients; as $2x^2y$ and $-7x^2y$.

Dissimilar or **Unlike Terms** are those which are not similar; as $3x^2y$ and $3xy^2$.

ADDITION OF MONOMIALS

33. The sum of a and b is expressed $a+b$ (§ 3).

34. We define **Subtraction**, in Algebra, as the process of finding one of two numbers, when their sum and the other number are given.

The **Minuend** is the sum of the numbers.

The **Subtrahend** is the given number.

The **Remainder** is the required number.

35. The remainder when b is subtracted from a is expressed $a-b$ (§ 4).

Since the sum of the remainder and the subtrahend gives the minuend (§ 34), we have

$$a-b+b=a.$$

Hence, *if the same number be both added to, and subtracted from, another, the value of the latter is not changed.*

36. It follows from § 35 that terms of equal absolute value, but opposite sign, in an expression, may be *cancelled.*

37. We will now show how to find the sum of a and $-b$.

By § 35, $$a+(-b)=a+(-b)+b-b; \qquad (1)$$

for adding and subtracting b does not alter $a+(-b)$.

But by § 20, $$(-b)+b=0;$$

for $-b$ and b are numbers of the same absolute value, but opposite sign.

Therefore, $$a+(-b)=a-b;$$

for the other terms in the second member of (1) cancel each other.

38. It follows from §§ 33 and 37 that the *addition of monomials is effected by uniting their terms with their respective signs.*

Thus, the sum of a, $-b$, c, $-d$, and $-e$ is

$$a-b+c-d-e.$$

39. We assume that the terms can be united *in any order,* provided each has its proper sign.

Hence, the result of § 38 can also be expressed

$$c+a-e-d-b,\ -d-b+c-e+a, \text{ etc.}$$

This law is called the *Commutative Law for Addition*; compare § 451.

40. To multiply $5+3$ by 4, we multiply 5 by 4, and then 3 by 4, and add the second result to the first.

Thus, $$(5+3)4=5\times 4+3\times 4.$$

We then assume that to multiply $a+b$ by c, we multiply a by c, and then b by c, and add the second result to the first.

Thus, $$(a+b)c=ac+bc.$$

This law is called the *Distributive Law for Multiplication*; its proof for the various forms of number will be found in § 455.

41. Addition of Similar Terms (§ 32).

1. Required the sum of $5a$ and $3a$.

We have,
$$\begin{aligned} 5a+3a &= (5+3)a && (\S\ 40)\\ &= 8a. \end{aligned}$$

2. Required the sum of $-5a$ and $-3a$.

We have,
$$\begin{aligned} (-5a)+(-3a) &= (-5)\times a+(-3)\times a && (\S\ 31)\\ &= [(-5)+(-3)]\times a && (\S\ 40)\\ &= (-8)\times a && (\S\ 20)\\ &= -8a. && (\S\ 31) \end{aligned}$$

3. Required the sum of $5a$ and $-3a$.

We have,
$$\begin{aligned} 5a+(-3)a &= [5+(-3)]\times a && (\S\ 40)\\ &= 2a. && (\S\ 20) \end{aligned}$$

4. Required the sum of $-5a$ and $3a$.

We have, $(-5)a+3a=[(-5)+3]\times a$ (§ 40)

$=(-2)\times a$ (§ 20) $=-2a$.

Therefore, *to add two similar terms, find the sum of their numerical coefficients* (§§ 20, 30, 31), *and affix to the result the common letters.*

5. Find the sum of $2a$, $-a$, $3a$, $-12a$, and $6a$.

Since the additions may be performed in any order, we may add the positive terms first, and then the negative terms, and finally combine these two results.

The sum of $2a$, $3a$, and $6a$ is $11a$.
The sum of $-a$ and $-12a$ is $-13a$.
Hence, the required sum is $11a+(-13a)$, or $-2a$.

6. Add $3(a-b)$, $-2(a-b)$, $6(a-b)$, and $-4(a-b)$.

The sum of $3(a-b)$ and $6(a-b)$ is $9(a-b)$.
The sum of $-2(a-b)$ and $-4(a-b)$ is $-6(a-b)$.
Then, the result is $[9+(-6)](a-b)$, or $3(a-b)$.

If the terms are not all similar, we may combine the similar terms, and unite the others with their respective signs (§ 38).

7. Required the sum of $12a$, $-5x$, $-3y^2$, $-5a$, $8x$, and $-3x$.

The sum of $12a$ and $-5a$ is $7a$.
The sum of $-5x$, $8x$, and $-3x$ is 0 (§ 36).
Then, the required sum is $7a-3y^2$.

EXERCISE 7

Add the following:

1. $11a$ and $-6a$.
2. $7x$ and $-10x$.
3. $-4n$ and $-9n$.
4. $-13ab$ and $5ab$.
5. $-17x^2$ and $-15x^2$.
6. $-abc$ and $12abc$.
7. $8x^3y^2$ and $-29x^3y^2$.
8. $9(a+b)$ and $-2(a+b)$.
9. $-17a^4mn^3$ and $60a^4mn^3$.
10. $8a$, $7a$, and $-9a$.

11. $15m$, $-m$, $-5m$, and $-12m$.

12. $16xyz$, $-4xyz$, xyz, and $-6xyz$.

13. $6(x-y)$, $-5(x-y)$, and $-10(x-y)$.

14. $18n^2$, $-13n^2$, $2n^2$, $-n^2$, and $-14n^2$.

15. $19a^3b$, $2a^3b$, $-3a^3b$, $-17a^3b$, and $10a^3b$.

16. $7ax$, $-9by$, $-3ax$, and $2by$.

17. $8x$, z, $-5y$, $-11z$, $-2x$, and $10y$.

18. $8(m+n)$, $4(m-n)$, $-3(m+n)$, and $-7(m-n)$.

19. $14a$, $-4d$, $-8c$, b, $-2a$, $-3c$, $-15d$, and $-c$.

20. $6x$, $-7y$, $5z$, $8y$, $-4z$, $-3x$, $-y$, $-9z$, and $2x$.

ADDITION OF POLYNOMIALS

42. A **Polynomial** is an algebraic expression consisting of more than one term; as $a+b$, or $2x^2-xy-3y^3$.

A polynomial is also called a *multinomial*.

A **Binomial** is a polynomial of *two* terms; as $a+b$.

A **Trinomial** is a polynomial of *three* terms; as $a+b-c$.

43. A polynomial is said to be *arranged* according to the *descending* powers of any letter, when the term containing the highest power of that letter is placed first, that having the next lower immediately after, and so on.

Thus, $$x^4+3x^3y-2x^2y^2+3xy^3-4y^4$$

is arranged according to the descending powers of x.

The term $-4y^4$, which does not involve x at all, is regarded as containing the lowest power of x in the above expression.

A polynomial is said to be arranged according to the *ascending* powers of any letter, when the term containing the lowest power of that letter is placed first, that having the next higher immediately after, and so on.

Thus, $$x^4+3x^3y-2x^2y^2+3xy^3-4y^4$$

is arranged according to the ascending powers of y.

44. Addition of Polynomials.

Let it be required to add $b+c$ to a.

Since $b+c$ is the sum of b and c (§ 3), we may add $b+c$ to a by adding b and c separately to a.

Then, $$a+(b+c)=a+b+c.$$

(To indicate the addition of $b+c$, we enclose it in parentheses.)

The above assumes that, to add the sum of a set of terms, we add the terms separately.

This is called the *Associative Law for Addition*; its proof will be found in § 452.

45. Let it be required to add $b-c$ to a.

By § 37, $b-c$ is the sum of b and $-c$.

Then, to add $b-c$ to a, we add b and $-c$ separately to a (§ 44).

Whence, $$a+(b-c)=a+b-c.$$

46. From §§ 44 and 45 we have the following rule:

To add a polynomial, add its terms with their signs unchanged.

1. Add $6a-7x^2$, $3x^2-2a+3y^3$, and $2x^2-a-mn$.

We set the expressions down one underneath the other, similar terms being in the same vertical column.

We then find the sum of the terms in each column, and write the results with their respective signs; thus,

$$\begin{array}{rrrr} 6a & -\,7x^2 & & \\ -\,2a & +\,3x^2 & +\,3y^3 & \\ -\,a & +\,2x^2 & & -\,mn \\ \hline 3a & -\,2x^2 & +\,3y^3 & -\,mn \end{array}$$

2. Add $4x-3x^2-11+5x^3$, $12x^2-7-8x^3-15x$, and $14+6x^3+10x-9x^2$.

It is convenient to arrange each expression in *descending* powers of x (§ 43); thus,

$$\begin{array}{rrrr} 5x^3 & -\,3x^2 & +\,4x & -\,11 \\ -\,8x^3 & +\,12x^2 & -\,15x & -\,7 \\ 6x^3 & -\,9x^2 & +\,10x & +\,14 \\ \hline 3x^3 & & -\,x & -\,4 \end{array}$$

3. Add $9(a+b)-8(b+c)$, $-3(b+c)-7(c+a)$, and $4(c+a)-5(a+b)$.

$$\begin{array}{rrr}
9(a+b) & -\ 8(b+c) & \\
 & -\ 3(b+c) & -\ 7(c+a) \\
-5(a+b) & & +\ 4(c+a) \\
\hline
4(a+b) & -\ 11(b+c) & -\ 3(c+a)
\end{array}$$

4. Add $\frac{3}{4}a+\frac{2}{5}b-\frac{1}{3}c$ and $\frac{1}{6}a-\frac{4}{3}b+\frac{5}{7}c$.

$$\begin{array}{rrr}
\frac{3}{4}a & +\ \frac{2}{5}b & -\ \frac{1}{3}c \\
\frac{1}{6}a & -\ \frac{4}{3}b & +\ \frac{5}{7}c \\
\hline
\frac{11}{12}a & -\ \frac{14}{15}b & +\ \frac{8}{21}c
\end{array}$$

EXERCISE 8

Add the following:

1.
$$\begin{array}{r}
8a-7b \\
-5a+4b \\
a-2b \\
\hline
\end{array}$$

2.
$$\begin{array}{r}
-\ 6x^3-16y^2 \\
9x^3+\ 3y^2 \\
-12x^3+10y^2 \\
\hline
\end{array}$$

3.
$$\begin{array}{r}
-17am+\ 4bn \\
6am-11bn \\
9am+19bn \\
\hline
\end{array}$$

4. $7x+6y-9z$ and $4x-8y+5z$.

5. $4m^2-4mn+n^2$, $m^2+4mn+4n^2$, and $-5m^2+5n^2$.

6. $5a-7b$, $4b-9c$, and $6c-2a$.

7. $3x^2-2xy+7y^2$, $-5x^2+9xy-10y^2$, and $8x^2-6xy-4y^2$.

8. $a-9-8a^2+16a^3$, $5+15a^3-12a-2a^2$, and $6a^2-10a^3+11a-13$.

9. $5(a+b)-4x(x-y)$, $-6(a+b)+3x(x-y)$, and $8(a+b)-7x(x-y)$.

10. $\frac{4}{3}a-\frac{1}{6}b-\frac{9}{14}c$ and $-\frac{3}{8}a+\frac{1}{2}b-\frac{5}{7}c$.

11. $5m+9n+4x$, $-3x-7y-6n$, $-10y+8x+2m$, and $n+11y-7m$.

12. $\frac{8}{15}x+\frac{2}{9}y+\frac{13}{10}z$ and $\frac{3}{10}x-\frac{7}{9}y-\frac{1}{2}z$.

13. $14(x+y)-17(y+z)$, $4(y+z)-9(z+x)$,
and $-3(x+y)-7(z+x)$.

14. $6c+2a-3b$, $4d-7c+12a$, $8b-5d+c$,
and $-10a-11b+9d$.

15. $-7(a-b)^2+8(a-b)+2$, $4(a-b)^2-5(a-b)$,
and $3(a-b)^2-9$.

16. $8a^3-11a-7a^2$, $2a-6a^2+10$, $-5+4a^3+9a$,
and $13a^2-5-12a^3$.

17. $x^2y+2xy^2+3x^3$, $3xy^2+4y^3-5x^2y$, $6x^3+5y^3-7xy^2$,
and $-8y^3+9x^2y-7x^3$.

18. $11x^2-13+4x^3+5x$, $-14x+2x^2+7+12x^3$,
$8x^3-3x-10+6x^2$, and $1-15x^2+9x-16x^3$.

19. $\frac{3}{2}a^2-\frac{4}{3}a-\frac{5}{4}$, $-\frac{1}{4}a^2+a+\frac{2}{5}$, and $-\frac{7}{8}a^2-\frac{6}{7}a+\frac{1}{2}$.

20. $5m^2n-n^3-4m^3+2mn^2$, $7mn^2-18m^2n+2m^3-9n^3$,
$-15mn^2+3m^2n+16n^3+8m^3$,
and $-5m^3+3mn^2-6n^3+10m^2n$.

21. $-5n^3+2n-12-15n^2$, $-14+7n-n^2-9n^3$,
$6n^2+13n^3+3-11n$, and $8-16n+10n^2+4n^3$.

SUBTRACTION OF MONOMIALS

47. The remainder when b is subtracted from a is expressed $a-b$ (§ 4).

We will now show how to subtract $-b$ from a.

By § 34, the sum of the remainder and the subtrahend equals the minuend.

Then, the required remainder must be an expression such that, when it is added to $-b$, the sum shall equal a.

But if $a+b$ is added to $-b$, the sum is a (§ 35).

Therefore, the required remainder is $a+b$.

That is, $$a-(-b)=a+b.$$

48. From § 47, we have the following rule:

To subtract a monomial, change its sign, and add the result to the minuend.

1. Subtract $5a$ from $2a$.

Changing the sign of the subtrahend, and adding the result to the minuend,

$$2a - 5a = 2a + (-5a) = -3a \text{ (§ 41).}$$

2. Subtract $-2a$ from $5a$.

$$5a - (-2a) = 5a + 2a = 7a.$$

3. Subtract $-5a$ from $-2a$.

$$-2a - (-5a) = -2a + 5a = 3a.$$

4. Subtract $5(x+y)$ from $-2(x+y)$.

$$-2(x+y) - 5(x+y) = -7(x+y).$$

The pupil should endeavor to put down the results, in examples like the above, without writing the intermediate step; changing the sign of the subtrahend *mentally*, and adding the result to the minuend.

5. From $-23a$ take the sum of $19a$ and $-5a$.

It is convenient to *change the sign of each expression which is to be subtracted*, and then add the results.

We then have $-23a - 19a + 5a$, or $-37a$.

EXERCISE 9

Subtract the following:

1. 9 from 3.
2. 2 from -6.
3. -16 from -10.
4. -5 from 12.
5. 42 from 15.
6. -28 from -61.
7. $-\frac{3}{4}$ from $\frac{1}{12}$.
8. $-\frac{13}{16}$ from $-\frac{3}{4}$.
9. $10\frac{5}{8}$ from $-3\frac{7}{8}$.

10.	11.	12.	13.	14.
$14a$	$4x$	$-6a^2$	$-15mn$	$-7x^2y$
$8a$	$-11x$	$4a^2$	$-7mn$	$-12x^2y$

15. $5bc$ from bc.
16. xyz from $-8xyz$.
17. $25a^2x^2$ from $13a^2x^2$.
18. $-40abc$ from $-23abc$.
19. $19(a-b)$ from $17(a-b)$.
20. $-18a^3bc^2$ from $-45a^3bc^2$.
21. From $7x$ take $-11y$.
22. From $-2a^3$ take $5n^2$.

23. From the sum of $18\,ab$ and $-9\,ab$ take the sum of $-21\,ab$ and $11\,ab$.

24. From the sum of $-13\,n^2$ and $24\,n^2$ take the sum of $46\,n^2$ and $-19\,n^2$.

25. From the sum of $16\,xy^3$ and $-37\,xy^3$ take the sum of $-29\,xy^3$, $34\,xy^3$, and $-47\,xy^3$.

SUBTRACTION OF POLYNOMIALS

49. Since a polynomial may be regarded as the sum of its separate terms (§ 38), we have the following rule:

To subtract a polynomial, change the sign of each of its terms, and add the result to the minuend.

1. Subtract $7\,ab^2 - 9\,a^2b + 8\,b^3$ from $5\,a^3 - 2\,a^2b + 4\,ab^2$.

It is convenient to place the subtrahend under the minuend, so that similar terms shall be in the same vertical column.

We then *mentally* change the sign of each term of the subtrahend, and add the result to the minuend; thus,

$$\begin{array}{rrrr}
5\,a^3 & -\,2\,a^2b & +\,4\,ab^2 & \\
 & -\,9\,a^2b & +\,7\,ab^2 & +\,8\,b^3 \\
\hline
5\,a^3 & +\,7\,a^2b & -\,3\,ab^2 & -\,8\,b^3
\end{array}$$

2. Subtract the sum of $9\,x^2 - 8\,x + x^3$ and $5 - x^2 + x$ from $6\,x^3 - 7\,x - 4$.

We change the sign of each expression which is to be subtracted, and add the results.

$$\begin{array}{rrrr}
6\,x^3 & & -\,7\,x & -\,4 \\
-\,x^3 & -\,9\,x^2 & +\,8\,x & \\
 & +\ \ x^2 & -\ \ x & -\,5 \\
\hline
5\,x^3 & -\,8\,x^2 & & -\,9
\end{array}$$

EXERCISE 10

Subtract the following:

1.
$$\begin{array}{r}
x^2 + 13\,x - 11 \\
\underline{-\,3\,x^2 + \ \ 6\,x - \ \ 5}
\end{array}$$

2.
$$\begin{array}{r}
-\,2\,m^2 - 4\,mn + \ \ 9\,n^2 \\
\underline{8\,m^2 - 7\,mn + 14\,n^2}
\end{array}$$

3.
$$\begin{array}{r}
ab + bc + ca \\
\underline{ab - bc + ca}
\end{array}$$

4. From $8x+2y-7z$ subtract $8x-2y+7z$.

5. From $4a^3-5a^2-15a-6$ take $a^3-12a^2-3a+11$.

6. From $7a-9c-b$ subtract $-5c+12a-8b$.

7. Subtract $-5(x+y)+9(x-y)$ from $7(x+y)-6(x-y)$.

8. Take $49x^2+16m^2-56mx$ from $25m^2+36x^2-60mx$.

9. By how much does $15x^3+6x^2y-4xy^2-11y^3$
exceed $8x^3-9x^2y+14xy^2-3y^3$?

10. Take $8a^3-12a^2b+6ab^2-b^3$
from $a^3-6a^2b+12ab^2-8b^3$.

11. What expression must be added to $3x^2-x+5$ to give 0?

12. By how much does $2m-4m^2-15+17m^3$
exceed $-9+6m^3-11m-14m^2$?

13. From $x+15x^3-18$ subtract $-2x^2-13+41x^3$.

14. Take $3b-16d+7a-10c$ from $-13c+14a-5d-9b$.

15. Subtract $12x-7n-6y$ from $11n+3m-8x$.

16. From $7n^2-5+20n^3+13n$ take $-9-14n^2+16n+5n^3$.

17. From $\frac{5}{6}a-\frac{1}{12}b+\frac{13}{16}c$ subtract $\frac{1}{2}a+\frac{2}{3}b-\frac{3}{8}c$.

18. Subtract $15a-21a^2+17$ from $-12a^2+22a^3-9a$.

19. Take $a^4-6a^3-15a^2-8a+4$
from $7a^4+3a^3-5a^2-11a-9$.

20. From $\frac{1}{2}m-\frac{1}{3}n+\frac{2}{7}p$ take $\frac{3}{5}m-\frac{3}{4}n+\frac{7}{8}p$.

21. From $n^4-10n^3x-n^2x^2+8nx^3+3x^4$
take $5n^4+4n^3x-9n^2x^2+2nx^3-12x^4$.

22. Take $18x^4-8x+6x^5+12-8x^3$
from $-10x^3+2-15x^2+11x^5-4x$.

23. Take $a^5-10a^3b^2+13a^2b^3-7ab^4-5b^5$
from $9a^5+3a^4b+6a^3b^2-a^2b^3-16b^5$.

24. From the sum of $2x^2-5xy-y^2$ and $7x^2-3xy+9y^2$
subtract $4x^2-6xy+8y^2$.

25. From 0 subtract the sum of $4a^2$ and $3a-5a^2-1$.

26. From $7x - 5z - 3y$ subtract the sum of $8y + 2x - 11z$ and $6z - 12y + 4x$.

27. From $6n^2 - 6n - 11$ subtract the sum of $2n^2 - 4n - 3$, $7n^2 - 10n + 4$, and $-3n^2 + 8n - 12$.

28. From the sum of $3b + 2a - 4c$ and $9c + 3b - 5d$ subtract the sum of $-6d - 7a$ and $8a - 7d + 9b + 5c$.

29. From the sum of $4a - 1 + 5a^3 - 8a^2$, $11 - 9a^2 + 3a^3 - 7a$, and $3a^2 - 7 + 10a - a^3$ subtract $-4a^3 + 9a - 6a^2 + 2$.

30. From the sum of $7x^3 - 4x^2 + 6x$ and $3x^2 - 10x - 5$ take the sum of $-5x^3 + 4x + 12$ and $8x^3 - 11x^2 - 2$.

PARENTHESES

50. Removal of Parentheses.

By § 45, $\qquad a + (b - c) = a + b - c.$

Hence, *parentheses preceded by a + sign may be removed without changing the signs of the terms enclosed.*

Again, by § 49, $\quad a - (b - c) = a - b + c.$

Hence, *parentheses preceded by a − sign may be removed if the sign of each term enclosed be changed, from + to −, and from − to +.*

The above rules apply equally to the removal of the *brackets, braces,* or *vinculum* (§ 12).

It should be noticed in the case of the latter that the sign apparently prefixed to the first term underneath is in reality prefixed to the vinculum; thus, $+\overline{a - b}$ means the same as $+(a - b)$, and $-\overline{a - b}$ the same as $-(a - b)$.

51. 1. Remove the parentheses from

$$2a - 3b - (5a - 4b) + (4a - b).$$

By the rules of § 50, the expression becomes

$$2a - 3b - 5a + 4b + 4a - b = a.$$

Parentheses sometimes enclose others; in this case they may be removed in succession by the rules of § 50.

Beginners should remove one at a time, commencing with the *innermost* pair; but after a little practice, they should be able to remove several signs of aggregation at one operation, in which case they should commence with the outermost pair.

2. Simplify $4x - \{3x + (-2x - \overline{x - a})\}$.

We remove the vinculum first, then the parentheses, and finally the braces.

Thus,

$$\begin{aligned} & 4x - \{3x + (-2x - \overline{x - a})\} \\ &= 4x - \{3x + (-2x - x + a)\} \\ &= 4x - \{3x - 2x - x + a\} \\ &= 4x - 3x + 2x + x - a = 4x - a. \end{aligned}$$

EXERCISE II

Simplify the following by removing the signs of aggregation, and then uniting similar terms:

1. $9m + (-4m + 6n) - (3m - n)$.
2. $2x - 3y - [5x + y] + \{-8x - 7y\}$.
3. $a - b - 2c + \overline{2a - b - c} - \overline{a - 2b - c}$.
4. $4y^2 - 2x^2 - [-4x^2 - 7xy + 5y^2] + (8x^2 - 9xy)$.
5. $3a^2 - 5ab - \{-4a^2 + 2ab - 9b^2\} - \overline{7a^2 - 6ab + b^2}$.
6. $5a - (7a - [9a + 4])$.
7. $7x - \{-8y - \overline{10x - 11y}\}$.
8. $6mn + 5 - ([-7mn - 3] - \{-5mn - 11\})$.
9. $8a^2 - 9 - (5a^2 - \overline{3a + 2}) + (6a^2 - \overline{4a - 7})$.
10. $2x - (8y + 5x - \overline{5x - y}) - (-9y + 3x)$.
11. $25 - (-8 - [-34 - \overline{16 - 47}])$.
12. $7x - (5x - [-12x + \overline{6x - 11}])$.
13. $2a - (-3b + c - \{a - b\}) - (3a + 2c - [-2b + 3c])$.

14. $5m-[7m-\{-3m-\overline{4m+9}\}-\overline{6m-8}]$.

15. $37-[41-\{13-(56-\overline{28+7})\}]$.

16. $9m-(3n+\{4m-[n-6m]\}-[m+7n])$.

17. $2a+[-6b-\{3c+(-4b-\overline{6c+a})\}]$.

18. $7x-(-6x-\{-5x-[-4x-\overline{3x-2}]\})$.

19. $5n-[8n-(3n+6)-\{-6n+\overline{7n-5}\}]$.

20. $4a-[a-\{-7a-(8a-\overline{5a+3})-(-6a-\overline{2a-9})\}]$.

21. $x-\{-11y-[2x-(-4y-\{-7x-5y\}-\overline{6x-9y})]\}$.

22. $3a-[b-(4b-7c)-\{2a-(3b-5c)-\overline{6b+c}\}]$.

23. $2x-[-4x-\{5x-(x-\overline{7x+6})\}+(3x-\overline{8x-9})]$.

52. Insertion of Parentheses.

To *enclose* terms in parentheses, we take the converse of the rules of § 50.

Any number of terms may be enclosed in parentheses preceded by a + sign, without changing their signs.

Any number of terms may be enclosed in parentheses preceded by a − sign, if the sign of each term be changed, from + to −, or from − to +.

Ex. Enclose the last three terms of $a-b+c-d+e$ in parentheses preceded by a − sign.

Result, $a-b-(-c+d-e)$.

EXERCISE 12

In each of the following expressions, enclose the last three terms in parentheses preceded by a − sign:

1. $a-b-c+d$.
2. m^3+2m^2+3m+4.
3. $x^3+x^2y-xy^2-y^3$.
4. $a^2-4b^2+12b-9$.
5. $4x^2-y^2-2yz-z^2$.
6. $a^2+b^2-c^2+d^2$.
7. $x^2-2xy+y^2+3x-4y$.
8. $n^4-5n^3-8n^2+6n+7$.

9. In each of the above results, enclose the last two terms in parentheses in brackets preceded by a $-$ sign.

53. Addition and Subtraction of Terms having Literal Coefficients.

To add two or more terms involving the same power of a certain letter, with literal, or numerical and literal, coefficients, it is convenient to put the coefficient of this letter in parentheses.

1. Add ax and $2x$.

By § 40, $$ax + 2x = (a + 2)x.$$

2. Add $(2m + n)y$ and $(m - 3n)y$.

$$(2m + n)y + (m - 3n)y = [(2m + n) + (m - 3n)]y$$
$$= (2m + n + m - 3n)y \text{(§ 50)} = (3m - 2n)y.$$

(The pupil should endeavor to put down the result in one operation.)

3. Subtract $(b - c)x^2$ from ax^2.

By § 48, $$ax^2 - (b - c)x^2 = [a - (b - c)]x^2$$
$$= (a - b + c)x^2 \text{ (§ 50)}.$$

EXERCISE 13

Add the following:

1. ax and bx.

2. mx^2 and $-2x^2$.

3. $-mny$ and $-pqy$.

4. mx, $-nx$, and $-px$.

5. a^2x^3 and $(ab - b^2)x^3$.

6. $(3a + 4b)n$ and $(5c - 7d)n$.

Subtract the following:

7. $2bx$ from $3ax$.

8. $-mny$ from aby.

9. $-nxy$ from $-axy$.

10. $(p + q)x$ from mx.

11. $(2a - 3b)y^2$ from $(5a - 4b)y^2$.

IV. MULTIPLICATION OF ALGEBRAIC EXPRESSIONS

54. The Rule of Signs.

If a and b are any two positive numbers, we have by § 25,

$$(+a)\times(+b)=+ab, \qquad (+a)\times(-b)=-ab,$$
$$(-a)\times(+b)=-ab, \qquad (-a)\times(-b)=+ab.$$

From these results we may state what is called the **Rule of Signs** in multiplication, as follows:

The product of two terms of like sign is positive; the product of two terms of unlike sign is negative.

55. We have by § 54,

$$(-a)\times(-b)\times(-c)=(ab)\times(-c)$$
$$=-abc; \qquad (1)$$
$$(-a)\times(-b)\times(-c)\times(-d)=(-abc)\times(-d), \text{ by (1),}$$
$$=abcd; \text{ etc.}$$

That is, the product of three negative terms is negative; the product of four negative terms is positive; and so on.

In general, *the product of any number of terms is positive or negative according as the number of negative terms is even or odd.*

56. The Law of Exponents.

Let it be required to multiply a^3 by a^2.

By § 11, $a^3=a\times a\times a,$

and $a^2=a\times a.$

Whence, $a^3\times a^2=a\times a\times a\times a\times a=a^5.$

We will now consider the general case.

Let it be required to multiply a^m by a^n, where m and n are any positive integers.

We have $a^m = a \times a \times \cdots$ to m factors,

and $a^n = a \times a \times \cdots$ to n factors.

Then, $a^m \times a^n = a \times a \times \cdots$ to $m+n$ factors $= a^{m+n}$.

(The *Sign of Continuation*, $\cdots$, is read "*and so on.*")

Hence, *the exponent of a letter in the product is equal to its exponent in the multiplicand plus its exponent in the multiplier.*

This is called the *Law of Exponents for Multiplication.*

A similar result holds for the product of three or more powers of the same letter.

Thus, $a^3 \times a^4 \times a^5 = a^{3+4+5} = a^{12}$.

MULTIPLICATION OF MONOMIALS

57. **1.** Let it be required to multiply $7a$ by $-2b$.

By § 31, $-2b = (-2) \times b$.

Then, $7a \times (-2b) = 7a \times (-2) \times b$

$= 7 \times (-2) \times a \times b = -14ab$. (§ 54)

In the above solution, we assume that *the factors of a product can be written in any order.*

This is called the *Commutative Law for Multiplication;* its proof for the various forms of number will be found in § 453.

2. Required the product of $-2a^2b^3$, $6ab^5$, and $-7a^4c$.

$$(-2a^2b^3) \times 6ab^5 \times (-7a^4c)$$
$$= (-2)a^2b^3 \times 6ab^5 \times (-7)a^4c$$
$$= (-2) \times 6 \times (-7) \times a^2 \times a \times a^4 \times b^3 \times b^5 \times c$$
$$= 84a^7b^8c, \text{ by §§ 55 and 56.}$$

We then have the following rule for the product of any number of monomials:

To the product of the numerical coefficients (§§ 30, 31, 55, 56) *annex the letters; giving to each an exponent equal to the sum of its exponents in the factors.*

3. Multiply $-5\,a^3b$ by $-8\,ab^3$.

$$(-5\,a^3b)\times(-8\,ab^3) = 40\,a^{3+1}b^{1+3} = 40\,a^4b^4.$$

4. Find the product of $4\,n^2$, $-3\,n^5$, and $2\,n^4$.

$$4\,n^2\times(-3\,n^5)\times 2\,n^4 = -24\,n^{2+5+4} = -24\,n^{11}.$$

5. Multiply $-x^m$ by $7\,x^6$.

$$(-x^m)\times 7\,x^6 = -7\,x^{m+6}.$$

6. Multiply $6\,(m+n)^4$ by $7\,(m+n)^3$.

$$6\,(m+n)^4\times 7\,(m+n)^3 = 42\,(m+n)^7.$$

EXERCISE 14

Multiply the following:

1. $9\,x^3$ by $4\,x^2$.
2. $-8\,a^4b$ by $7\,ab^4$.
3. $11\,ax$ by $-3\,by$.
4. $-7\,xy^6$ by $-9\,x^5y$.
5. $15\,b^4c^5$ by $2\,a^2b^3$.
6. $-x^my^nz^r$ by x^3y^2z.
7. $13(x-y)$ by $-(x-y)^2$.
8. $-5\,a^5b^8c^7$ by $-12\,a^6b^4c^3$.
9. $9(a+b)^5$ by $6(a+b)^3$.
10. $-6\,a^5x^6y^9$ by $11\,x^8y^7z^4$.
11. $-2\,a^{2m}b^{3n}$ by $-5\,a^mb^{4n}$.
12. $14\,x^py^{2m}$ by $-8\,x^qy^n$.
13. $4\,m^3$, $-7\,m^5$, and $-3\,m^7$.
14. $2\,a^4$, $6\,b^5$, and $-8\,c^6$.
15. $a^{3p}b^q$, b^4c^m, and c^na^{2p}.
16. $-5\,x^9y^6$, $-9\,y^3z^5$, and $-z^7x$.
17. $2\,x^2$, $-x^6$, $6\,x^4$, and $4\,x^8$.
18. $-3\,a^6b$, $-5\,b^5c$, $-2\,c^3a$, and $-a^2b^4c^7$.
19. $3\,m^2n^3x^2$, $-4\,m^3n^4y^5$, $-5\,m^4x^5y^2$, and $6\,n^5x^4y^3$.
20. $a^{2m}c^q$, $-b^{3p}d^r$, $-a^nd^2$, and $-b^pc$.
21. m^8n^5, $-2\,mx^6$, $3\,m^5y^3$, $-5\,n^4x^2$, and $-4\,n^6y^4$.

MULTIPLICATION OF POLYNOMIALS BY MONOMIALS

58. In § 40, we assumed that the product of $a+b$ by c was $ac+bc$.

We then have the following rule for the product of a polynomial by a monomial:

Multiply each term of the multiplicand by the multiplier, and add the partial products.

Ex. Multiply $2x^2-5x+7$ by $-8x^3$.

$$\begin{aligned}(2x^2-5x+7)\times(-8x^3)\\ &=(2x^2)\times(-8x^3)+(-5x)\times(-8x^3)+(7)\times(-8x^3)\\ &=-16x^5+40x^4-56x^3.\end{aligned}$$

The student should put down the final result in one operation.

EXERCISE 15

Multiply the following:

1. $5x-12$ by $7x$.
2. $10a^3b+7ab^4$ by $-6ab^3$.
3. $x^4-4x^2y^2+4y^4$ by $-x^2y^2$.
4. $8m^4-m^2-3$ by $5m^4$.
5. $8x^5$ by $6x^3+5x-17$.
6. $-4a^3b^3$ by $3a^2-2ab-4b^2$.
7. $7x^my^{4n}-8x^5y^p$ by $-3x^6y^n$.
8. $6a^5-4a^7-5a^6$ by $9a^7$.
9. $-m^2n+8n^2-3m^4$ by $-12m^5n^2$.
10. $2a^4b^4$ by $a^3-6a^2b+12ab^2-8b^3$.
11. $9n^2-6-2n^3-5n+n^4$ by $-3n^6$.
12. $5x^3-2x^2y+4xy^2-3y^3$ by $11xy$.

MULTIPLICATION OF POLYNOMIALS BY POLYNOMIALS

59. Let it be required to multiply $a+b$ by $c+d$.

As in § 40, we multiply $a+b$ by c, and then $a+b$ by d, and add the second result to the first; that is,

$$\begin{aligned}(a+b)(c+d)&=(a+b)c+(a+b)d\\ &=ac+bc+ad+bd.\end{aligned}$$

We then have the following rule:

Multiply each term of the multiplicand by each term of the multiplier, and add the partial products.

60. **1.** Multiply $3a-4b$ by $2a-5b$.

In accordance with the rule, we multiply $3a-4b$ by $2a$, and then by $-5b$, and add the partial products.

A convenient arrangement of the work is shown below, similar terms being in the same vertical column.

$$\begin{array}{lll}
3a & -4b & \\
2a & -5b & \\
\hline
6a^2 & -\ 8ab & \\
 & -15ab & +20b^2 \\
\hline
6a^2 & -23ab & +20b^2
\end{array}$$

The work may be *verified* by performing the example with the multiplicand and multiplier interchanged.

2. Multiply $4ax^2+a^3-8x^3-2a^2x$ by $2x+a$.

It is convenient to arrange the multiplicand and multiplier in the same order of powers of some common letter (§ 43), and write the partial products in the same order.

Arranging the expressions according to the descending powers of a, we have

$$\begin{array}{llllll}
a^3 & -2a^2x & +4ax^2 & -8x^3 & & \\
a & +2x & & & & \\
\hline
a^4 & -2a^3x & +4a^2x^2 & -8ax^3 & & \\
 & \quad 2a^3x & -4a^2x^2 & +8ax^3 & -16x^4 & \\
\hline
a^4 & & & & -16x^4 &
\end{array}$$

EXERCISE 16

Multiply the following:

1. $5x-7$ by $3x+2$.
2. $8m+n$ by $8m+n$.
3. $2a-3$ by $6a-7$.
4. $-10xy+8$ by $-5xy-4$.
5. m^2-m-3 by $m+3$.
6. a^2-a-12 by $a-7$.
7. $4(a-b)-3$ by $4(a-b)+3$.
8. $x^2-2xy+3y^2$ by $x-3y$.
9. $4m^2+9n^2-6mn$ by $3n+2m$.
10. $\frac{1}{3}a-\frac{1}{4}b$ by $\frac{1}{3}a-\frac{1}{4}b$.
11. $x-4y$ by $x^2+4xy+16y^2$.

12. $a+b+c$ by $a-b-c$.

13. $5m^2+3m-4$ by $6m^3+5m^2$.

14. $8-4n+2n^2-n^3$ by $2+n$.

15. $2a^2-3a+5$ by a^2+a-2.

16. $6(m+n)^2-5(m+n)+1$ by $7(m+n)-2$.

17. $2x^3-3x^2-5x-1$ by $3x-5$.

18. $6x+2x^2+8$ by $-4+x^2-3x$.

19. $2n^2+m^2+3mn$ by $2n^2-3mn+m^2$.

20. $\frac{9}{4}x^2-\frac{3}{5}x+\frac{4}{25}$ by $\frac{3}{2}x+\frac{2}{5}$.

21. $4a^2+6a-10$ by $2a^2-3a+5$.

22. $9x+2x^2-5$ by $4+3x^2-7x$.

23. $10n^2+3n-4$ by $9n^2-5n-6$.

24. $x^{2p+6}y-x^7y^q$ by $x^{2p-1}+y^{q-1}$.

25. $a^3+2a^2b+2ab^2+b^3$ by $a^2-2ab+b^2$.

26. $m^4-3m^3+9m^2-27m+81$ by $m+3$.

27. $3(a+b)^2-2(a+b)+1$ by $4(a+b)^2-(a+b)+5$.

28. $3+a^3-7a-4a^2$ by $a+a^2-7$.

29. $8m^3+12m^2n+18mn^2+27n^3$ by $2m^2n-3mn^2$.

30. $4a^{m+5}b^2-3a^4b^n$ by $a^{m+2}b-2ab^{n-1}$.

31. $-a^3-2a^2+6a-5$ by $a^2-2a+10$.

32. $5x^4-6x^3-4x^2+2x-3$ by $3x-2$.

33. $4m^3+6m^2n-5mn^2-3n^3$ by $3m^2+2mn-n^2$.

34. $a^{2n+4}+a^{n+2}b^{2n+1}+b^{4n+2}$ by $a^{n+3}b-ab^{2n+2}$.

35. $mx+my-nx-ny$ by $mx-my+nx-ny$.

36. $a^3-3a^2x+3ax^2-x^3$ by $a^3+3a^2x+3ax^2+x^3$.

37. $x^2-6xy+9y^2$ by $x^3-9x^2y+27xy^2-27y^3$.

38. $a^m+b^n-c^p$ by $a^m-b^n+c^p$.

39. $2n^3 - 3n^2 - n - 4$ by $2n^3 - 3n^2 + n - 4$.

40. $5x^2 - 7 + 2x^3 - 8x$ by $-4 + 3x^2 - 5x$.

41. $5a^4 + a^3 - 2a^2 - 6a + 3$ by $2a^2 - a - 6$.

42. $\frac{2}{3}m^2 - \frac{3}{4}m - 1$ by $\frac{1}{2}m^2 + \frac{1}{3}m - \frac{4}{9}$.

43. $a + 3$, $a + 4$, and $a - 5$.

44. $x - 6$, $3x - 2$, and $4x + 1$.

45. $m + 2n$, $m^2 - 2mn + 4n^2$, and $m^3 - 8n^3$.

46. $4m - 7$, $5m - 8$, and $6m - 5$.

47. $x + 2$, $x - 3$, $x - 5$, and $x + 6$.

48. $a + 2b$, $3a - 4b$, and $3a^2 - 2ab - 8b^2$.

49. $2x + y$, $2x - y$, $4x^2 + y^2$, and $16x^4 + y^4$.

50. $2m + 3n$, $2m - 3n$, $3m + 2n$, and $3m - 2n$.

51. $n^2 + n + 2$, $n^2 - n + 2$, and $n^4 + 3n^2 - 4$.

52. $a - 2$, $a + 3$, $3a - 1$, and $3a^3 - 2a^2 - 19a - 6$.

61. If the product has more than one term involving the same power of a certain letter, with literal, or numerical and literal, coefficients, we put the coefficient of this letter in parentheses, as in § 53.

Ex. Multiply $x^2 - ax - bx + ab$ by $x - a$.

$$\begin{array}{l}
x^2 - ax \quad - \;\; bx \;\; + ab \\
\underline{x \;\; - a \qquad\qquad\qquad\qquad\qquad\qquad} \\
x^3 - ax^2 \;\; - \;\; bx^2 \qquad\qquad + abx \\
\underline{\quad\; - ax^2 \qquad\qquad\quad + a^2x + abx - a^2b} \\
x^3 - (2a + b)x^2 + (a^2 + 2ab)x - a^2b
\end{array}$$

As in § 53, $-2ax^2 - bx^2$ is equivalent to $-(2a + b)x^2$, and $a^2x + 2abx$ to $(a^2 + 2ab)x$.

EXERCISE 17

Multiply the following:

1. $x^2 + ax + bx + ab$ by $x + a$.

2. $x^2 - mx + nx - mn$ by $x - p$.
3. $x^2 - bx - cx + bc$ by $x - a$.
4. $x^2 + ax - bx - 3ab$ by $x + b$.
5. $x^2 + ax + 2bx + 2ab$ by $x - c$.
6. $x^2 + px - 5qx - 5pq$ by $x - r$.
7. $x^2 - 3ax - bx + 3ab$ by $x + 2a$.
8. $x^2 - 4mx + nx - 4mn$ by $x + 3n$.
9. $x^2 - 3ax - 2bx + 6ab$ by $x - 4c$.
10. $(a - b)x - 3ab$ by $2x - (a - b)$.
11. $x^{2n} - 5ax^n + 4bx^n - 2ab$ by $x^n + c$.
12. $(2a - 1)x^2 + (a + 2)x - (a + 3)$ by $(a - 2)x - a$.

62. *Ex.* Simplify $(a - 2x)^2 - 2(3a + x)(a - x)$.

To simplify the expression, we first multiply $a - 2x$ by itself (§ 11); we then find the product of 2, $3a + x$, and $a - x$, and subtract the second result from the first.

$$\begin{array}{lll} a & -2x & \\ a & -2x & \\ \hline a^2 & -2ax & \\ & -2ax & +4x^2 \\ \hline a^2 & -4ax & +4x^2 \end{array} \qquad\qquad \begin{array}{lll} 3a & +x & \\ a & -x & \\ \hline 3a^2 & +ax & \\ & -3ax & -x^2 \\ \hline 3a^2 & -2ax & -x^2 \\ & & 2 \\ \hline 6a^2 & -4ax & -2x^2 \end{array}$$

Subtracting the second result from the first, we have

$$a^2 - 4ax + 4x^2 - 6a^2 + 4ax + 2x^2 = -5a^2 + 6x^2.$$

EXERCISE 18

Simplify the following:

1. $(3a + 5)(2a - 8) + (4a - 7)(a + 6)$.
2. $(3x + 2)(4x + 3) - (3x - 2)(4x - 3)$.
3. $(a - 2x)(b + 3y) + (a + 2x)(b - 3y)$.

4. $(3m+1)^2(3m-1)^2$.

5. $(x-y)(x^2-y^2)-(x+y)(x^2+y^2)$.

6. $(2a+3b)^2-4(a-b)(a+5b)$.

7. $[3x-(5y+2z)][3x-(5y-2z)]$.

8. $[m+2n-(2m-n)][2m+n-(m-2n)]$.

9. $(a+b+c)^2-(a-b-c)^2$.

10. $(a+2)(a+3)(a-4)+(a-2)(a-3)(a+4)$.

11. $(\frac{5}{2}x-\frac{4}{3}y+\frac{1}{4}z)^2$.

12. $[2x^2+(3x-1)(4x+5)][5x^2-(4x+3)(x-2)]$.

13. $(a+2b-c-3d)^2$.

14. $(a-2)(a+3)-(a-3)(a+4)-(a-4)(a+5)$.

15. $(x+2)(2x-1)(3x-4)-(x-2)(2x+1)(3x+4)$.

16. $[x-(z-y)][y-(x-z)][z-(x-y)]$.

17. $(a-b)(a^3+b^3)[a(a+b)+b^2]$.

18. $(a+b-2c)^2-(b+c-2a)^2+(c+a-2b)^2$.

19. $(x+y+z)^2+(x-y-z)^2+(-x+y-z)^2+(-x-y+z)^2$.

20. $(2x+1)^3+(2x-1)^3$.

21. $(a+b+c)(ab+bc+ca)-(a+b)(b+c)(c+a)$.

22. $(a+2b)^2-2(a+2b)(2a+b)+(2a+b)^2$.

23. $(x+y+z)^3-3(y+z)(z+x)(x+y)$.

24. $(a+b)^3+3(a+b)^2(a-b)+3(a+b)(a-b)^2+(a-b)^3$.

DEFINITIONS

63. A monomial is said to be *rational and integral* when it is either a number expressed in Arabic numerals, or a single letter with unity for its exponent, or the product of two or more such numbers or letters.

Thus, $3a^2b^3$, being equivalent to $3\cdot a\cdot a\cdot b\cdot b\cdot b$, is rational and integral.

A polynomial is said to be rational and integral when each term is rational and integral; as $2x^2 - \frac{3}{4}ab + c^3$.

64. If a term has a literal portion which consists of a single letter with unity for its exponent, the term is said to be of the *first degree*.

Thus, $2a$ is of the first degree.

The *degree* of any rational and integral monomial (§ 63) is the number of terms of the first degree which are multiplied together to form its literal portion.

Thus, $5ab$ is of the *second* degree; $3a^2b^3$, being equivalent to $3 \cdot a \cdot a \cdot b \cdot b \cdot b$, is of the *fifth* degree; etc.

The degree of a rational and integral monomial equals the sum of the exponents of the letters involved in it.

Thus, ab^4c^3 is of the *eighth* degree.

The degree of a rational and integral polynomial is the degree of its term of highest degree.

Thus, $2a^2b - 3c + d^2$ is of the *third* degree.

65. Homogeneity.

Homogeneous terms are terms of the same degree.

Thus, a^4, $3b^3c$, and $-5x^2y^2$ are homogeneous terms.

A polynomial is said to be homogeneous when its terms are homogeneous; as $a^3 + 3b^2c - 4xyz$.

66. If the multiplicand and multiplier are homogeneous, the product will also be homogeneous, and its degree equal to the sum of the degrees of the multiplicand and multiplier.

The examples in § 60 are instances of the above law; thus, in Ex. 2, the multiplicand, multiplier, and product are homogeneous, and of the third, first, and fourth degrees, respectively.

The student should always, when possible, apply the principles of homogeneity to test the accuracy of algebraic work.

Thus, if two homogeneous expressions be multiplied together, and the product obtained is not homogeneous, it is evident that the work is not correct.

V. DIVISION OF ALGEBRAIC EXPRESSIONS

67. We define **Division**, in Algebra, as the process of finding one of two numbers, when their product and the other number are given.

The **Dividend** is the product of the numbers.

The **Divisor** is the given number.

The **Quotient** is the required number.

68. The Rule of Signs.

Since the dividend is the product of the divisor and quotient, the equations of § 54 may be written as follows:

$$\frac{+ab}{+a} = +b, \quad \frac{-ab}{-a} = +b, \quad \frac{-ab}{+a} = -b, \text{ and } \frac{+ab}{-a} = -b.$$

From these results, we may state the **Rule of Signs** in division, as follows:

The quotient of two terms of like sign is positive; the quotient of two terms of unlike sign is negative.

69. Let $$\frac{a}{b} = x. \qquad (1)$$

Then, since the dividend is the product of the divisor and quotient, we have $$a = bx.$$

Multiply each of these equals by c (Ax. 7, § 9),

$$ac = bcx.$$

Regarding ac as the dividend, bc as the divisor, and x as the quotient, this may be written

$$\frac{ac}{bc} = x. \qquad (2)$$

From (1) and (2), $$\frac{ac}{bc} = \frac{a}{b}. \quad \text{(Ax. 4, § 9)} \qquad (3)$$

That is, *a factor common to the dividend and divisor can be removed, or cancelled.*

70. The Law of Exponents for Division.

Let it be required to divide a^5 by a^2.

By § 11, $$\frac{a^5}{a^2}=\frac{a\times a\times a\times a\times a}{a\times a}.$$

Cancelling the common factor $a\times a$ (§ 69), we have

$$\frac{a^5}{a^2}=a\times a\times a=a^3.$$

We will now consider the general case.

Let it be required to divide a^m by a^n, where m and n are any positive integers such that m is greater than n.

We have, $$\frac{a^m}{a^n}=\frac{a\times a\times a\times\cdots\text{ to } m \text{ factors}}{a\times a\times a\times\cdots\text{ to } n \text{ factors}}.$$

Cancelling the common factor $a\times a\times a\times\cdots$ to n factors,

$$\frac{a^m}{a^n}=a\times a\times a\times\cdots\text{ to } m-n \text{ factors}=a^{m-n}.$$

Hence, *the exponent of a letter in the quotient is equal to its exponent in the dividend, minus its exponent in the divisor.*

This is called the *Law of Exponents for Division.*

DIVISION OF MONOMIALS

71. 1. Let it be required to divide $-14a^2b$ by $7a^2$.

By § 57, $$\frac{-14a^2b}{7a^2}=\frac{(-2)\times 7\times a^2\times b}{7\times a^2}.$$

Cancelling the common factors 7 and a^2 (§ 69), we have

$$\frac{-14a^2b}{7a^2}=(-2)\times b=-2b.$$

Then to find the quotient of two monomials:

To the quotient of the numerical coefficients annex the letters, giving to each an exponent equal to its exponent in the dividend minus its exponent in the divisor, and omitting any letter having the same exponent in the dividend and divisor.

2. Divide $54\,a^5b^3c^2$ by $-9\,a^4b^3$.

$$\frac{54\,a^5b^3c^2}{-9\,a^4b^3} = -6\,a^{5-4}c^2 = -6\,ac^2.$$

3. Divide $-2\,x^{2m}y^nz^r$ by $-x^my^nz^5$.

$$\frac{-2\,x^{2m}y^nz^r}{-x^my^nz^5} = 2\,x^{2m-m}z^{r-5} = 2\,x^mz^{r-5}.$$

4. Divide $35\,(a-b)^7$ by $7\,(a-b)^4$.

$$\frac{35\,(a-b)^7}{7\,(a-b)^4} = 5\,(a-b)^3.$$

EXERCISE 19

Divide the following:

1. 30 by -5.
2. -42 by 6.
3. -48 by -4.
4. -64 by 8.
5. -135 by -9.
6. 176 by -11.
7. $-\frac{12}{35}$ by $\frac{8}{15}$.
8. $21\,a^{10}$ by $3\,a^7$.
9. $-63\,m^4n^6$ by $7\,m^2n^3$.
10. $6\,x^6y^{10}$ by $-x^5y^{10}$.
11. $9\,(a-b)^5$ by $3\,(a-b)^2$.
12. xy^2z^3 by $-xyz$.
13. $-13\,m^{10}n^6$ by $-13\,m^5n^4$.
14. $45\,(x+y)^7$ by $-5\,(x+y)^3$.
15. $72\,x^7y^8$ by $6\,x^7y^8$.
16. $-40\,a^2b^4c^5$ by $-8\,bc$.
17. $90\,a^{11}x^8$ by $9\,a^7x^7$.
18. $-28\,a^3b^9c^5$ by $2\,b^8c^5$.
19. $a^{m+1}b^{n+3}$ by $-ab^2$.
20. $-55\,x^5y^9z^{10}$ by $-11\,y^2z^3$.
21. $-70\,a^7b^8c^9$ by $14\,ab^6c^4$.
22. $-32\,x^{5p}y^{q+r}$ by $-8\,x^{2p}y^r$.
23. $-96\,x^{2m-1}y^n$ by $12\,x^{m+3}y^p$.
24. $52\,a^{12}b^6c^{13}$ by $-4\,a^5bc^3$.
25. $132\,x^4y^4z^{14}$ by $12\,x^4y^2z^5$.

Find the numerical value when $a=2$, $b=-4$, $c=5$, and $d=-3$ of:

26. $\dfrac{10\,ab}{cd}+\dfrac{8\,ac}{bd}$.

27. $\dfrac{2\,a-b}{c-5\,d}-\dfrac{a+4\,b}{3\,c+d}$.

28. $\dfrac{7\,a+14\,b-12\,c}{13\,a-9\,b+17\,c}$.

29. $\dfrac{a-b}{a+3\,b}+\dfrac{b-c}{b+5\,c}-\dfrac{c-d}{c+4\,d}$.

DIVISION OF POLYNOMIALS BY MONOMIALS

72. We have, $(a+b)c = ac + bc.$

Since the dividend is the product of the divisor and quotient (§ 67), we may regard $ac + bc$ as the dividend, c as the divisor, and $a + b$ as the quotient.

Whence, $$\frac{ac+bc}{c} = a + b.$$

Hence, *to divide a polynomial by a monomial, we divide each term of the dividend by the divisor, and add the results.*

Ex. Divide $9\,a^2b^2 - 6\,a^4c + 12\,a^3bc^3$ by $-3\,a^2$.

$$\frac{9\,a^2b^2 - 6\,a^4c + 12\,a^3bc^3}{-3\,a^2} = -3\,b^2 + 2\,a^2c - 4\,abc^3.$$

EXERCISE 20

Divide the following:

1. $25\,a^8 - 15\,a^6 + 40\,a^4$ by $5\,a^4$.
2. $-24\,m^5n^2 + 33\,mn^7$ by $-3\,mn^2$.
3. $36\,x^7yz^6 - 9\,x^5y^3z^2 - 27\,x^3y^2z^4$ by $9\,x^3y$.
4. $54\,a^4b^5c^6 - 60\,a^7b^6c^2$ by $6\,ab^5c^2$.
5. $-22\,x^{10}y^3 + 30\,x^7y^6 + 26\,x^4y^9$ by $-2\,x^4y^3$.
6. $70\,n^{13} - 56\,n^{11} - 63\,n^9 + 49\,n^7$ by $7\,n^7$.
7. $66\,x^4yz + 77\,xy^6z - 55\,xyz^4$ by $-11\,xyz$.
8. $36\,a^{15} + 28\,a^{12} - 4\,a^9 - 20\,a^6$ by $4\,a^6$.
9. $x^{p+q}y^{n+2} - x^{q+1}y^4$ by x^qy^3.
10. $14\,m^5n^2 - 28\,m^4n^3 + 28\,m^3n^4 - 14\,m^2n^5$ by $-14\,m^2n^2$.
11. $32\,x^{15} + 24\,x^{13} - 48\,x^{11} - 40\,x^9$ by $-8\,x^8$.
12. $84\,x^5y^8z^6 - 108\,x^4y^7z^8 - 48\,x^9y^5z^7$ by $12\,x^4y^5z^6$.
13. $a^{7p}b^qc^{4r} - a^{5p}b^{3q}c^{2r} - a^{3p}b^{5q}c^{3r}$ by $-a^{3p}b^qc^{2r}$.
14. $30\,a^{10}m^5n^2 - 60\,a^7m^8n^9 - 45\,a^9mn^8$ by $-15\,a^7mn^2$.

DIVISION OF POLYNOMIALS BY POLYNOMIALS

73. Let it be required to divide $12+10x^3-11x-21x^2$ by $2x^2-4-3x$.

Arranging each expression according to the descending powers of x (§ 43), we are to find an expression which, when multiplied by the divisor, $2x^2-3x-4$, will produce the dividend, $10x^3-21x^2-11x+12$.

It is evident that the term containing the highest power of x in the product is the product of the terms containing the highest powers of x in the multiplicand and multiplier.

Therefore, $10x^3$ is the product of $2x^2$ and the term containing the highest power of x in the quotient.

Whence, the term containing the highest power of x in the quotient is $10x^3$ divided by $2x^2$, or $5x$.

Multiplying the divisor by $5x$, we have the product $10x^3-15x^2-20x$; which, when subtracted from the dividend, leaves the remainder $-6x^2+9x+12$.

This remainder must be the product of the divisor by the rest of the quotient; therefore, to obtain the next term of the quotient, we regard $-6x^2+9x+12$ as a new dividend.

Dividing the term containing the highest power of x, $-6x^2$, by the term containing the highest power of x in the divisor, $2x^2$, we obtain -3 as the second term of the quotient.

Multiplying the divisor by -3, we have the product $-6x^2+9x+12$; which, when subtracted from the second dividend, leaves no remainder.

Hence, $5x-3$ is the required quotient.

$$
\begin{array}{rrrr|l}
10x^3 & -21x^2 & -11x & +12 & 2x^2-3x-4, \text{ Divisor.} \\
\cline{5-5}
10x^3 & -15x^2 & -20x & & 5x-3, \text{ Quotient.} \\
\cline{1-4}
 & -6x^2 & +9x & +12 & \\
 & -6x^2 & +9x & +12 & \\
\cline{2-4}
\end{array}
$$

The example might have been solved by arranging the dividend and divisor according to the *ascending* powers of x.

From the above example, we derive the following rule.

Arrange the dividend and divisor in the same order of powers of some common letter.

Divide the first term of the dividend by the first term of the divisor, and write the result as the first term of the quotient.

Multiply the whole divisor by the first term of the quotient, and subtract the product from the dividend.

If there be a remainder, regard it as a new dividend, and proceed as before; arranging the remainder in the same order of powers as the dividend and divisor.

1. Divide $9ab^2 + a^3 - 9b^3 - 5a^2b$ by $3b^2 + a^2 - 2ab$.

Arranging according to the descending powers of a,

$$
\begin{array}{l|l}
a^3 - 5a^2b + 9ab^2 - 9b^3 & a^2 - 2ab + 3b^2 \\ \cline{2-2}
\underline{a^3 - 2a^2b + 3ab^2} & a - 3b \\
\quad - 3a^2b + 6ab^2 & \\
\quad \underline{- 3a^2b + 6ab^2 - 9b^3} &
\end{array}
$$

In the above example, the last term of the second dividend is omitted, as it is merely a repetition of the term directly above.

The work may be verified by multiplying the quotient by the divisor, which should of course give the dividend.

2. Divide $4 + 9x^4 - 28x^2$ by $-3x^2 + 2 + 4x$.

Arranging according to the ascending powers of x,

$$
\begin{array}{l|l}
4 - 28x^2 + 9x^4 & 2 + 4x - 3x^2 \\ \cline{2-2}
\underline{4 + 8x - 6x^2} & 2 - 4x - 3x^2 \\
\quad - 8x - 22x^2 + 9x^4 & \\
\quad \underline{- 8x - 16x^2 + 12x^3} & \\
\qquad\quad - 6x^2 - 12x^3 + 9x^4 & \\
\qquad\quad \underline{- 6x^2 - 12x^3 + 9x^4} &
\end{array}
$$

EXERCISE 21

Divide the following:

1. $6a^2 + 29a + 35$ by $2a + 5$.
2. $30x^2 - 53x + 8$ by $6x - 1$.
3. $32x^2 + 28xy - 15y^2$ by $4x + 5y$.
4. $10a^3 + 33a^2 - 52a + 9$ by $5a - 1$.

5. $a^3 - 8b^3$ by $a - 2b$.

6. $28n^2 + 34n - 12$ by $-7n + 2$.

7. $64x^3 + 27y^3$ by $4x + 3y$.

8. $6(x-y)^2 - 7(x-y) - 20$ by $3(x-y) + 4$.

9. $25m^2n^2 - 36$ by $6 + 5mn$. 5mn − 6

10. $-25x^3y + 12x^4 + 12x^2y^2$ by $3x^2 - 4xy$.

11. $18m^3 - 17mx^2 - 6x^3$ by $3m + 2x$.

12. $2n^2 - 6 + 5n^3 - 19n$ by $-8n + 5n^2 - 3$.

13. $12 + 13x^2 - 19x - 12x^3$ by $-3x^2 - 4 + x$.

14. $x^2 - y^2 - 2yz - z^2$ by $x - y - z$.

15. $8(a+b)^3 - c^3$ by $2(a+b) - c$.

16. $8m^4n - 24m^2n^3 + 12m^5 - 31m^3n^2$ by $6m^2 - 8n^2 - 5mn$.

17. $8a + 9a^4 - 1 - 16a^2$ by $1 - 3a^2 - 4a$.

18. $\frac{8}{9}m^2 - \frac{1}{3}m - \frac{3}{4}$ by $\frac{2}{3}m + \frac{1}{2}$.

19. $2x^3 - 10 - 6x^2 + x^4 + 11x$ by $2 + x^2 - x$.

20. $x^4 - 81$ by $x^3 - 3x^2 + 9x - 27$.

21. $a^4 - 256b^4$ by $a - 4b$.

22. $m^4 - 7m^2n^2 + n^4$ by $m^2 + 3mn + n^2$.

23. $81x^4 - 1$ by $1 + 3x$.

24. $6 - a^3 + 6a^4 - 8a - 23a^2$ by $2a + 3$.

25. $(x+y)^3 - 9(x+y)^2 + 27(x+y) - 27$ by $(x+y) - 3$.

26. $-36x^2 - 1 + 4x^4 - 12x$ by $-2x^2 + 1 + 6x$.

27. $10a - a^2 - 25 + 16a^4$ by $5 + 4a^2 - a$.

28. $\frac{8}{27}x^3 + \frac{27}{64}$ by $\frac{2}{3}x + \frac{3}{4}$.

29. $3n^4 - 11n^3 - 25n^2 - 13n - 2$ by $3n^2 + 4n + 1$.

30. $2x^2y^2 + y^4 + 9x^4$ by $-2xy + 3x^2 + y^2$.

31. $73x + 37x^3 - 35 + 20x^4 - 15x^2$ by $-5 + 4x^2 + 9x$.

32. $243\,n^5 + 1$ by $3\,n + 1$.

33. $x^4 + 16\,x^3 + 96\,x^2 + 256\,x + 256$ by $(x + 4)^2$.

34. $-60\,n^3 + 127\,n^2 + 214\,n - 336$ by $-12\,n^2 + 11\,n + 56$.

35. $-32 + a^5$ by $8\,a + a^4 + 16 + 2\,a^3 + 4\,a^2$.

36. $a^{2m+1}b^{2n} + 2\,a^{2m+3}b^{2n+1} + a^{2m+5}b^{2n+2}$ by $a^m b^{n-1} + a^{m+2}b^n$.

37. $\frac{1}{27}\,a^3 - \frac{1}{12}\,a^2b + \frac{1}{16}\,ab^2 - \frac{1}{64}\,b^3$ by $\frac{1}{3}\,a - \frac{1}{4}\,b$.

38. $5\,x^3 + 24 - 33\,x^2 + 10\,x + 3\,x^4$ by $-4 + 3\,x$.

39. $x^5 + 37\,x^2 - 70\,x + 50$ by $x^2 - 2\,x + 10$.

40. $x^{3m+2} + 8\,x^{3m-1}$ by $x^{2m+1} - 2\,x^{2m} + 4\,x^{2m-1}$.

41. $(3\,a^2 + 5\,a - 2)(2\,a^2 - a - 6)$ by $(3\,a - 1)(2\,a + 3)$.

42. $63\,x^4 + 114\,x^3 + 49\,x^2 - 16\,x - 20$ by $9\,x^2 + 6\,x - 5$.

43. $a^{3p+1}b^2 - ab^{3q+2}$ by $a^{p+1}b^2 - ab^{q+2}$.

44. $a^5 - b^5 - 5\,a^4b + 5\,ab^4 + 10\,a^3b^2 - 10\,a^2b^3$
by $a^3 - b^3 - 3\,a^2b + 3\,ab^2$.

45. $-6\,n^5 - 25\,n^4 + 7\,n^3 + 81\,n^2 + 3\,n - 28$
by $-2\,n^3 - 5\,n^2 + 8\,n + 7$.

46. $23\,x^2 - 5\,x^4 - 12 + 12\,x^5 + 8\,x - 14\,x^3$ by $x - 2 + 3\,x^2$.

47. $15\,a^3 - 4\,a^5 - 15 + 8\,a^6 - 5\,a - 2\,a^4 + 3\,a^2$ by $-a + 4\,a^3 - 3$.

48. $52\,x^3 + 64 + 18\,x^4 - 200\,x^2 + x^5$ by $6\,x^2 - 8 + x^3 - 12\,x$.

49. $a^{2m} - b^{2n} - 2\,b^nc^p - c^{2p}$ by $a^m - b^n - c^p$.

50. $a^6 - 6\,a^4n^2 + 9\,a^2n^4 - 4\,n^6$ by $a^3 - 2\,a^2n - an^2 + 2\,n^3$.

51. $3\,x^7 - 7\,x^5 - 11\,x^4 + 5\,x^3 + 7\,x^2 + 5\,x - 2$ by $3\,x^3 - x - 2$.

52. $5\,n^3 + 6\,n - n^5 - 8 + 6\,n^6 + 46\,n^2 - 38\,n^4$
by $2 - 5\,n^2 + 3\,n^3 - 4\,n$.

53. $\frac{4}{9}\,m^4 - 2\,m^3 + \frac{9}{4}\,m^2 - \frac{1}{16}$ by $\frac{2}{3}\,m^2 - \frac{3}{2}\,m - \frac{1}{4}$.

54. $9\,x^2 - 25\,y^2 - 40\,yz - 16\,z^2$ by $3\,x + 5\,y + 4\,z$.

55. $90\,n^4 - 143\,n^3 - 102\,n^2 + 131\,n + 60$ by $(2\,n - 3)(5\,n + 4)$.

56. $x^{4m} + x^{2m}y^{4n} + y^{8n}$ by $x^{2m} - x^my^{2n} + y^{4n}$.

57. $4x^6-21x^4y^2+21x^2y^4-4y^6$ by $(x-y)(x-2y)(2x+y)$.

58. $a^5+32+10a(a^3+8)+40a^2(a+2)$ by $(a-2)^2+8a$.

74. By § 66, if the dividend and divisor are *homogeneous*, the quotient will be homogeneous, and its degree equal to the degree of the dividend minus the degree of the divisor.

75. The operation of division is often facilitated by the use of parentheses.

Ex. Divide $x^3+(a+b-c)x^2+(ab-bc-ca)x-abc$ by $x+a$.

$$\begin{array}{l|l}
x^3+(a+b-c)x^2+(ab-bc-ca)x-abc & x+a \\
\underline{x^3+\qquad ax^2} & x^2+(b-c)x-bc \\
\qquad (b-c)x^2 & \\
\qquad \underline{(b-c)x^2+(ab\qquad -ca)x} & \\
\qquad\qquad\qquad\quad -\ bcx & \\
\qquad\qquad\qquad\quad \underline{-\ bcx-abc} &
\end{array}$$

EXERCISE 22

Divide the following:

1. $x^3+(a-b-c)x^2+(-ab+bc-ca)x+abc$
   by $x^2+(a-b)x-ab$.
2. $x^3+(a+b-c)x^2+(ab-bc-ca)x-abc$ by $x-c$.
3. $x^3-(a+b+c)x^2+(ab+bc+ca)x-abc$ by $x^2-(b+c)x+bc$.
4. $x^3-(a-2b-3c)x^2+(-2ab+6bc-3ca)x-6abc$
   by $x^2-(a-3c)x-3ac$.
5. $x^3+(3a+b+2c)x^2+(3ab+2bc+6ca)x+6abc$ by $x+3a$.
6. $a(a-b)x^2+(-ab+b^2+bc)x-c(b+c)$ by $(a-b)x+c$.
7. $m(m+n)x^2-(m^2+n^2)x+n(m-n)$ by $mx-n$.
8. $x^2-(m-2n)x-2m^2+11mn-15n^2$ by $x+m-3n$.
9. $(2m^2+10mn)x^2+(8m^2-9mn-15n^2)x-(12mn-9n^2)$
   by $2mx-3n$.
10. $x^3-(3a+2b-4c)x^2+(6ab-8bc+12ca)x-24abc$ by $x-2b$.

VI. INTEGRAL LINEAR EQUATIONS

76. Any term of either member of an equation is called a *term* of the equation.

77. A **Numerical Equation** is one in which all the known numbers are represented by Arabic numerals; as,

$$2x-7=x+6.$$

An **Integral Equation** is one each of whose members is a rational and integral expression (§ 63); as,

$$4x-5=\tfrac{2}{3}y+1.$$

78. An **Identical Equation**, or **Identity**, is one whose members are equal, whatever values are given to the letters involved; as $(a+b)(a-b)=a^2-b^2$.

The sign $\equiv$, read "*is identically equal to*," is frequently used in place of the sign of equality in an identity.

79. An equation is said to be *satisfied* by a set of values of certain letters involved in it when, on substituting the value of each letter in place of the letter wherever it occurs, the equation becomes identical.

Thus, the equation $x-y=5$ is satisfied by the set of values $x=8$, $y=3$; for, on substituting 8 for x, and 3 for y, the equation becomes $8-3=5$, or $5=5$; which is identical.

80. An **Equation of Condition** is an equation involving one or more letters, called **Unknown Numbers**, which is satisfied only by particular values of these letters.

Thus, the equation $x+2=5$ is not satisfied by every value of x, but only by the particular value $x=3$.

An equation of condition is usually called an *equation*.

Any letter in an equation of condition may represent an unknown number; but it is usual to represent unknown numbers by the last letters of the alphabet.

81. If an equation contains but one unknown number, any value of the unknown number which satisfies the equation is called a **Root** of the equation.

Thus, 3 is a root of the equation $x+2=5$.

To *solve* an equation is to find its roots.

82. If a rational and integral monomial (§ 63) involves a certain letter, its *degree with respect to it* is denoted by its exponent.

If it involves two letters, its *degree with respect to them* is denoted by the sum of their exponents; etc.

Thus, $2ab^4x^2y^3$ is of the *second* degree with respect to x, and of the *fifth* with respect to x and y.

83. If an integral equation (§ 77) contains one or more unknown numbers, the *degree* of the equation is the degree of its term of highest degree.

Thus, if x and y represent unknown numbers,

$ax-by=c$ is an equation of the *first* degree;

$x^2+4x=-2$, an equation of the *second* degree;

$2x^2-3xy^2=5$, an equation of the *third* degree; etc.

A **Linear**, or **Simple, Equation** is an equation of the first degree.

PRINCIPLES USED IN SOLVING INTEGRAL EQUATIONS

84. Since the members of an equation are equal numbers, we may write the last four axioms of § 9 as follows:

1. *The same number, or equal numbers, may be added to both members of an equation without destroying the equality.*

2. *The same number, or equal numbers, may be subtracted from both members of an equation without destroying the equality.*

3. *Both members of an equation may be multiplied by the same number, or equal numbers, without destroying the equality.*

4. *Both members of an equation may be divided by the same number, or equal numbers, without destroying the equality.*

85. Transposing Terms.

Consider the equation $x+a-b=c$.

Adding $-a$ and $+b$ to both members (§ 84, 1), we have

$$x=c-a+b.$$

In this case, the terms $+a$ and $-b$ are said to be *transposed* from the first member to the second.

Hence, *any term may be transposed from one member of an equation to the other by changing its sign.*

86. It follows from § 85 that

If the same term occurs in both members of an equation affected with the same sign, it may be cancelled.

87. Consider the equation

$$a-x=b-c. \qquad (1)$$

Multiplying each term by -1 (§ 84), we have

$$x-a=c-b;$$

which is the same as equation (1) with the sign of every term changed.

Hence, *the signs of all the terms of an equation may be changed, without destroying the equality.*

88. Clearing of Fractions.

Consider the equation

$$\frac{2}{3}x-\frac{5}{4}=\frac{5}{6}x-\frac{9}{8}.$$

Multiplying each term by 24, the lowest common multiple of the denominators (Ax. 7, § 9), we have

$$16x-30=20x-27,$$

where the denominators have been removed.

Removing the fractions from an equation by multiplication is called *clearing the equation of fractions.*

SOLUTION OF INTEGRAL LINEAR EQUATIONS

89. To solve an equation involving one unknown number, we put it into a succession of forms, which finally leads to the value of the root.

This process is called *transforming* the equation.

Every transformation is effected by means of the principles of §§ 84 to 88, inclusive.

90. Examples.

1. Solve the equation

$$5x-7=3x+1.$$

Transposing $3x$ to the first member, and -7 to the second (§ 85), we have

$$5x-3x=7+1.$$

Uniting similar terms, $2x=8.$

Dividing both members by 2 (§ 84, 4),

$$x=4.$$

To *verify* the result, put $x=4$ in the given equation.

Thus, $20-7=12+1$; which is identical.

2. Solve the equation

$$\frac{7}{6}x-\frac{5}{3}=\frac{3}{5}x-\frac{1}{4}.$$

Clearing of fractions by multiplying each term by 60, the L. C. M. of 6, 3, 5, and 4, we have

$$70x-100=36x-15.$$

Transposing $36x$ to the first member, and -100 to the second,

$$70x-36x=100-15.$$

Uniting terms, $34x=85.$

Divided by 34, $x=\frac{85}{34}=\frac{5}{2}.$

3. Solve the equation

$$(5-3x)(3+4x)=62-(7-3x)(1-4x).$$

Expanding, $15 + 11x - 12x^2 = 62 - (7 - 31x + 12x^2).$

Or, $15 + 11x - 12x^2 = 62 - 7 + 31x - 12x^2.$

Cancelling the $-12x^2$ terms (§ 86), and transposing,

$$11x - 31x = 62 - 7 - 15.$$

Uniting terms, $-20x = 40.$

Dividing by -20, $x = -2.$

To *expand* an algebraic expression is to perform the operations indicated.

From the above examples, we have the following rule for solving an integral linear equation with one unknown number:

Clear the equation of fractions, if any, by multiplying each term by the L. C. M. *of the denominators of the fractional coefficients.*

Remove the parentheses, if any, by performing all the operations indicated.

Transpose the unknown terms to the first member, and the known to the second; cancelling any term which has the same coefficient in both members.

Unite similar terms, and divide both members by the coefficient of the unknown number.

The pupil should verify every result.

EXERCISE 23

Solve the following equations, in each case verifying the answer:

1. $8x + 7 = 95.$
2. $9x = 5x - 32.$
3. $7x + 15 = 2x + 45.$
4. $10x - 3 = 3x - 38.$
5. $6x + 13 = 11x - 7.$
6. $5 - 18x = 83 - 12x.$
7. $11x - 3 = 4 + 3x.$
8. $6x - 28 = 15x - 13.$
9. $19 - 13x = 31 - 29x.$
10. $14x - 51 = 27x - 33.$
11. $13 + 12x = 37x + 43.$
12. $21x - 23 = 51 - 16x.$
13. $11x + 17 = 65x + 47.$
14. $98 - 16x = 23 - 41x.$

15. $17x-9+47=41x-35x+27.$

16. $13x-39=48x-29x-81.$

17. $54=26x-31x+19x-9.$

18. $2x-\frac{5}{3}x+\frac{1}{7}x=10.$

19. $\frac{5}{4}x+\frac{4}{5}x=-\frac{41}{10}.$

20. $\frac{5}{2}x+\frac{1}{6}x+\frac{8}{3}x=-\frac{16}{9}.$

21. $\frac{5}{6}x=\frac{7}{4}x-\frac{3}{8}x+\frac{13}{6}.$

22. $\frac{4}{7}x+\frac{1}{4}=\frac{9}{14}x+\frac{3}{28}x.$

23. $\frac{2}{5}x-\frac{38}{15}=\frac{8}{9}x-\frac{4}{3}x.$

24. $\frac{5}{6}x-\frac{7}{8}x=\frac{7}{9}x-\frac{1}{18}x-\frac{55}{48}.$

25. $4(2x-7)+5=5(x-3)+16.$

26. $13x-(5x-8)=6x-(3x+7).$

27. $80-6(4x+3)=7x-3(6x+1).$

28. $x-2(4-7x)=4x-9(2-3x).$

29. $8x(3x+2)-27=4x(6x-1)-147.$

30. $45-5x(6x-1)=21-3x(10x+3).$

31. $4(x+14)-4(3x-32)=6(x+12)-7(x-12).$

32. $(5-3x)(3+4x)=(7+3x)(1-4x)-1.$

33. $(1+3x)^2=(5-x)^2+4(1-x)(3-2x).$

34. $6(x+4)^2=5-(2x+3)^2-5(2-x)(7+2x).$

35. $(3x-2)^2-9(x-1)(3x-8)=-18x^2+51x-38.$

36. $(x+4)^3-(x-4)^3=2(3x-2)(4x+1).$

37. $\frac{1}{3}\left(4x+1\right)+\frac{1}{5}\left(6x-2\right)-\frac{1}{6}\left(5x+8\right)=2.$

38. $\frac{5}{6}\left(x-3\right)-\frac{2}{9}\left(2x+7\right)-\frac{7}{12}\left(x-5\right)=\frac{2}{9}.$

39. $\frac{1}{3}\left(4+x\right)-\frac{7}{12}\left(1-5x\right)=\frac{3}{8}\left(1+2x\right)-\frac{5}{16}\left(2-3x\right).$

PROBLEMS LEADING TO INTEGRAL LINEAR EQUATIONS WITH ONE UNKNOWN NUMBER

91. For the solution of a problem by algebraic methods, the following suggestions will be found of service:

1. Represent the unknown number, or one of the unknown numbers if there are several, by some letter, as x.

2. Every problem contains, explicitly or implicitly, *just as many distinct statements as there are unknown numbers involved.*

Use all but one of these to express the other unknown numbers in terms of x.

3. Use the remaining statement to form an equation.

92. Problems.

1. Divide 45 into two parts such that the less part shall be one-fourth the greater.

Here there are *two* unknown numbers; the greater part and the less.

In accordance with the first suggestion of § 91, we represent the greater part by x.

The first statement of the problem is, implicitly:

The sum of the greater part and the less is 45.

The second statement is:

The less part is one-fourth the greater.

In accordance with the second suggestion of § 91, we use the *first statement* to express the less part in terms of x.

Thus, the less part is represented by $45 - x$.

We now, in accordance with the third suggestion, use the *second statement* to form an equation.

Thus, $$45 - x = \frac{1}{4}x.$$

Clearing of fractions, $$180 - 4x = x.$$

Transposing, $$-4x - x = -180, \text{ or } -5x = -180.$$

Dividing by -5, $$x = 36, \text{ the greater part.}$$

Then, $$45 - x = 9, \text{ the less part.}$$

2. A had twice as much money as B; but after giving B \$28, he has $\frac{4}{5}$ as much as B. How much had each at first?

Let x represent the number of dollars B had at first.

Then, $2x$ will represent the number A had at first.

Now after giving B \$28, A has $2x-28$ dollars, and B $x+28$ dollars; we then have the equation

$$2x-28=\frac{2}{3}(x+28).$$

Clearing of fractions, $6x-84=2(x+28)$.

Expanding, $6x-84=2x+56$.

Transposing, $4x=140$.

Dividing by 4, $x=35$, the number of dollars B had at first;

and $2x=70$, the number of dollars A had at first.

3. A is 3 times as old as B, and 8 years ago he was 7 times as old as B. Required their ages at present.

Let $x=$ the number of years in B's age.

Then, $3x=$ the number of years in A's age.

Also, $x-8=$ the number of years in B's age 8 years ago,

and $3x-8=$ the number of years in A's age 8 years ago.

But A's age 8 years ago was 7 times B's age 8 years ago.

Whence, $3x-8=7(x-8)$.

Expanding, $3x-8=7x-56$.

Transposing, $-4x=-48$.

Dividing by -4, $x=12$, the number of years in B's age.

Whence, $3x=36$, the number of years in A's age.

4. A sum of money amounting to \$4.32 consists of 108 coins, all dimes and cents; how many are there of each kind?

Let $x=$ the number of dimes.

Then, $108-x=$ the number of cents.

Also, the x dimes are worth $10x$ cents.

But the entire sum amounts to 432 *cents*.

Whence, $10x+108-x=432$.

Transposing, $9x=324$.

Whence, $x=36$, the number of dimes;

and $108-x=72$, the number of cents.

EXERCISE 24

1. The difference of two numbers is 12, and 7 times the smaller exceeds the greater by 30. Find the numbers.

2. The sum of two numbers is 29, and the smaller exceeds their difference by 4. Find the numbers.

3. Find two numbers whose sum is $\frac{7}{8}$, and difference $\frac{1}{8}$.

4. The sum of two numbers is 44, and their difference is three-fourths the smaller number. Find the numbers.

5. A is 4 times as old as B; and in 22 years he will be twice as old. Find their ages.

6. A is 3 times as old as B; and $6\frac{1}{2}$ years ago he was 5 times as old. Find their ages.

7. A has 3 times as much money as B; but after B gives him $9, he has 6 times as much as B. How much had each at first?

8. A man has 21 coins, all dimes and twenty-five-cent pieces, valued in all at $3.30. How many has he of each?

9. A is 25 years of age, and B is 16. In how many years will B be two-thirds as old as A?

10. Divide 43 into two parts such that if the greater be added to 17, and the less to 30, the resulting numbers shall be equal.

11. Twice a certain number exceeds 35 by the same amount that one-third the number exceeds 5. Find the number.

12. Divide $280 between A, B, and C so that A's share may exceed $\frac{2}{5}$ of B's by $96, and B's share exceed C's by $20.

13. A is 22 years of age, and B is 18. How many years ago was A's age $\frac{8}{5}$ of B's?

14. A man has $4.10, all five-cent and fifty-cent pieces; and he has 5 more five-cent than fifty-cent pieces. How many has he of each?

15. The sum of $\frac{4}{9}$ and $\frac{3}{2}$ a certain number exceeds $\frac{5}{6}$ the number by $\frac{2}{3}$. Find the number.

16. If A has \$5.50, and B \$3.50, how much money must A give B in order that B may have $\frac{4}{5}$ as much as A?

17. A room is $\frac{6}{5}$ as long as it is wide; if the width were increased by $1\frac{1}{2}$ feet, and the length diminished by the same amount, the room would be square. Find its dimensions.

18. The sum of two numbers is $\frac{13}{10}$ the greater, and their difference is $\frac{7}{4}$. Find the numbers.

19. A boy buys a certain number of apples at 2 for 5 cents, and double the number at 3 for 5 cents, and spent in all 35 cents. How many of each kind did he buy?

20. Divide \$320 between A, B, C, and D so that A may receive \$35 more than B, C \$15 more than B, and D \$25 less than C.

21. The sum of the ages of A, B, and C is 52 years; A's age is $\frac{4}{3}$ of B's, and A is 8 years younger than C. Find their ages.

22. In a certain school the boys are 15 fewer than $\frac{2}{3}$ of the whole, and the girls are 33 more than $\frac{1}{5}$. How many boys, and how many girls, are there?

23. The sum of \$900 is invested, part at 4%, and the rest at 5%, per annum, and the total annual income is \$42. How much is invested in each way?

24. In 9 years B will be $\frac{5}{6}$ as old as A; and 12 years ago he was $\frac{3}{5}$ as old. What are their ages?

Let x represent the number of years in A's age 12 years ago.

25. A has $\frac{2}{5}$ of a certain sum of money, B has $\frac{1}{3}$, C has $\frac{2}{9}$, and D has the remainder, \$8. How much have A, B, and C?

26. A man bought 8 hens, 7 sheep, and 12 pigs for \$269; each sheep cost $1\frac{1}{8}$ as much as each hen, and \$3 less than each pig. What did each cost?

27. Divide 66 into two parts such that $\frac{6}{7}$ the greater shall exceed $\frac{5}{8}$ the less by 21.

28. Find two numbers whose sum is 10, such that the square of the greater exceeds the square of the less by 40.

29. Find two consecutive numbers such that $\frac{1}{5}$ the greater exceeds $\frac{1}{8}$ the less by 2.

30. A person attempting to arrange a certain number of counters in a square finds that he has too few by 12; but on reducing the number in the side of the square by 3, he has 21 left over. How many has he?

31. A purse contains a certain number of 10-shilling pieces, twice as many 5-shilling pieces, and 5 times as many shillings, the contents of the purse being worth £5. How many are there of each coin?

32. The square of the third of three consecutive numbers exceeds the product of the other two by 13. Find the numbers.

33. Divide 39 into two parts such that 3 times the smaller shall be as much below 58 as twice the greater exceeds 38.

34. Find two numbers whose difference is 3, and whose product is less by 33 than the square of the greater.

35. The total number of persons at a certain factory is 196; the number of women is $\frac{3}{2}$ the number of men, and $\frac{5}{8}$ the number of boys. How many of each are there?

36. A room is twice as long as it is wide, and it is found that 50 square feet of carpet, 1 foot in width, are required to make a border around it. Find its dimensions.

37. A purse contains a certain number of dimes, $\frac{7}{8}$ as many cents, and $\frac{1}{4}$ as many \$1 bills, the value of the entire contents being \$5.74. How many are there of each?

38. A starts to walk from P to Q, 12 miles, at the same time that B starts to walk from Q to P. They meet at the end of 2 hours. If A walks one mile an hour faster than B, what are their rates?

39. Divide \$210 between A, B, C, and D so that B may receive \$10 less than A, C $\frac{11}{7}$ as much as B, and D $\frac{5}{8}$ as much as A.

40. The sum of $32 is divided between 7 men, 8 women, and 16 children; each child receiving $\frac{1}{4}$ as much as each man, and each woman 75 cents more than each child. How much is received by each man, each woman, and each child?

41. A boy had a certain number of marbles. He lost 6 of them, gave away $\frac{1}{2}$ the remainder, and then found that he had 5 more than $\frac{1}{3}$ of his original number. How many had he at first?

42. There are two heaps of coins, one containing 5-cent pieces and the other 10-cent pieces. The second heap is worth 20 cents more than the first, and has 8 fewer coins. Find the number in each heap.

43. In an audience of 435 persons, there are 25 more women than men, and 3 times as many girls as men; and the number of boys is less by 195 than twice the number of girls. Find the number of each.

44. Find four consecutive odd numbers such that the product of the first and third shall be less than the product of the second and fourth by 64.

45. A sum of money, amounting to $19.30, consists of $2 bills, 25-cent pieces, and 5-cent pieces. There are 13 more 5-cent pieces than $2 bills, and $\frac{7}{8}$ as many 5-cent pieces as 25-cent pieces. Find the number of each.

46. Two barrels contain 46 and 45 gallons of water, respectively. A certain number of gallons are drawn from the first, and $\frac{3}{4}$ as many from the second, and the second now contains $\frac{3}{2}$ as many gallons as the first. How many gallons were drawn from each?

47. A tank containing 150 gallons can be filled by one pipe in 15 minutes, and emptied by another in 25 minutes. After the first pipe has been open a certain number of minutes, it is closed, and the second pipe opened; and the tank is emptied in 24 minutes from the time the first pipe was opened. How many minutes is each pipe open?

VII. SPECIAL METHODS IN MULTIPLICATION AND DIVISION

93. Any Power of a Power.

Required the value of $(a^2)^3$.

By § 11, $(a^2)^3 = a^2 \times a^2 \times a^2 = a^6$.

We will now consider the general case:

Required the value of $(a^m)^n$, where m and n are any positive integers.

We have, $(a^m)^n = a^m \times a^m \times \cdots$ to n factors

$$= a^{m+m+\cdots \text{ to } n \text{ terms}} = a^{mn}.$$

94. Any Power of a Product.

Required the value of $(ab)^3$.

By § 11, $(ab)^3 = ab \times ab \times ab = a^3b^3$.

We will now consider the general case:

Required the value of $(ab)^n$, where n is any positive integer.

We have, $(ab)^n = ab \times ab \times \cdots$ to n factors $= a^nb^n$.

In like manner, $(abc\cdots)^n = a^nb^nc^n\cdots$,

whatever the number of factors in $abc\cdots$.

95. Any Power of a Monomial.

1. Find the value of $(-5a^4)^3$.

By § 31, $(-5a^4)^3 = [(-5) \times a^4]^3$

$$= (-5)^3 \times (a^4)^3 (\S\ 94) = -125a^{12} (\S\ 93).$$

2. Find the value of $(-2m^3n)^4$.

We have, $(-2m^3n)^4 = (-2)^4 \times (m^3)^4 \times n^4 = 16m^{12}n^4$.

96. From §§ 93 and 94 and the examples of § 95, we have the following rule for raising a rational and integral monomial (§ 63) to any power whose exponent is a positive integer:

Raise the absolute value of the numerical coefficient to the required power, and multiply the exponent of each letter by the exponent of the required power.

Give to every power of a positive term, and to every **even** *power of a negative term, the positive sign; and to every* **odd** *power of a negative term the negative sign.*

EXERCISE 25

Expand the following:

1. $(x^3y^4z^5)^8$.	5. $(7\,a^mb^{2n})^3$.	9. $(a^nb^3c)^p$.
2. $(m^6n^2p)^{11}$.	6. $(-n^8x^5y^4)^{10}$.	10. $(x^{5m}y^{4n}z^{3p})^{12}$.
3. $(-ab^7c^{10})^7$.	7. $(2\,m^6x^7)^6$.	11. $(-3\,m^2n^9x^6)^5$.
4. $(-11\,x^9y^8)^2$.	8. $(-4\,x^2yz^{11})^4$.	12. $(-2\,am^5n^7)^9$.

97. Square of a Binomial.

Let it be required to square $a+b$.

$$\begin{array}{r} a+b \\ a+b \\ \hline a^2+ab \qquad \\ ab+b^2 \\ \hline \end{array}$$

Whence, $$(a+b)^2 = a^2 + 2\,ab + b^2. \qquad (1)$$

That is, *the square of the sum of two numbers equals the square of the first, plus twice the product of* ~~*the first by the second*~~, *plus the square of the second.*

1. Square $3\,a+2\,b$.

We have, $(3\,a+2\,b)^2 = (3\,a)^2 + 2(3\,a)(2\,b) + (2\,b)^2$

$$= 9\,a^2 + 12\,ab + 4\,b^2.$$

Let it be required to square $a-b$.

$$\begin{array}{l} a-b \\ a-b \\ \hline a^2 \quad -ab \\ \qquad -ab+b^2. \\ \hline \end{array}$$

Whence, $$(a-b)^2 = a^2 - 2\,ab + b^2. \tag{2}$$

That is, *the square of the difference of two numbers equals the square of the first, minus twice the product of the ~~first by the sec-ond~~, plus the square of the second.* two

In the remainder of the work we shall use the expression "the *difference* of a and b" to denote the *remainder obtained by subtracting b from a.*

The result (2) may also be derived by substituting $-b$ for b, in equation (1).

2. Square $4\,x^2 - 5$.

We have, $$(4\,x^2-5)^2 = (4\,x^2)^2 - 2(4\,x^2)(5) + 5^2$$
$$= 16\,x^4 - 40\,x^2 + 25.$$

If the first term of the binomial is *negative*, it should be enclosed, negative sign and all, in parentheses, before applying the rules.

3. Square $-2\,a^3+9$.

We have, $$(-2\,a^3+9)^2 = [(-2\,a^3)+9]^2$$
$$= (-2\,a^3)^2 + 2(-2\,a^3)(9) + 9^2$$
$$= 4\,a^6 - 36\,a^3 + 81.$$

EXERCISE 26

Expand the following:

1. $(a+2)^2$.
2. $(x-5)^2$.
3. $(6\,x-7\,y)^2$.
4. $(3+8\,n^2)^2$.
5. $(-m^4+4\,p^5)^2$.
6. $(9\,ab-1)^2$.
7. $(7\,x+3\,x^2)^2$.
8. $(n^3+11\,n^4)^2$.
9. $(2\,a^3-7\,b^2c)^2$.
10. $(-4\,m^4-3\,n^3)^2$.
11. $(6\,x^5y+x^2y^4)^2$.
12. $(5\,ab+8\,bc)^2$.
13. $(4\,x^3-11\,yz)^2$.
14. $(5\,ax-12\,by)^2$.
15. $(-3\,n^7+10\,n^5)^2$.
16. $(8\,x^5+9\,x^6)^2$.
17. $(7\,a^5m^2-13\,b^4n^3)^2$.
18. $(-6\,xy-11\,xz)^2$.
19. $(5\,x^p+4\,y^{2q})^2$.
20. $(2\,a^m-9\,a^n)^2$.

What is the

98. Product of the Sum and Difference of Two Numbers.

Let it be required to multiply $a+b$ by $a-b$.

$$\begin{array}{r} a+b \\ a-b \\ \hline a^2+ab \qquad\quad \\ -ab-b^2 \\ \hline \end{array}$$

Whence, $(a+b)(a-b)=a^2-b^2$.

That is, *the product of the sum and difference of two numbers equals the difference of their squares.*

1. Multiply $6a+5b^3$ by $6a-5b^3$.

By the rule,

$$(6a+5b^3)(6a-5b^3)=(6a)^2-(5b^3)^2=36a^2-25b^6.$$

2. Multiply $-x^2+4$ by $-x^2-4$.

$$\begin{aligned}(-x^2+4)(-x^2-4)&=[(-x^2)+4][(-x^2)-4]\\&=(-x^2)^2-4^2=x^4-16.\end{aligned}$$

3. Expand $(x+y+z)(x-y+z)$.

$$\begin{aligned}(x+y+z)(x-y+z)&=[(x+z)+y][(x+z)-y]\\&=(x+z)^2-y^2\\&=x^2+2xz+z^2-y^2.\end{aligned}$$

4. Expand $(a+b-c)(a-b+c)$.

By § 52, $(a+b-c)(a-b+c)=[a+(b-c)][a-(b-c)]$

$$\begin{aligned}&=a^2-(b-c)^2, \text{ by the rule,}\\&=a^2-(b^2-2bc+c^2)\\&=a^2-b^2+2bc-c^2.\end{aligned}$$

EXERCISE 27

Expand the following:

1. $(7a+2b)(7a-2b)$.
2. $(9m^2+4)(9m^2-4)$.
3. $(3x^3+8yz^2)(3x^3-8yz^2)$.
4. ([illegible] a^3+6)([illegible] a^3-6).

5. $(11m^5+5n^4)(11m^5-5n^4)$. 7. $(5a^2+12b^3c)(5a^2-12b^3c)$.

6. $(4x^5y+7z^6)(4x^5y-7z^6)$. 8. $(a^{2m}+x^{3n})(a^{2m}-x^{3n})$.

9. $(10m^4n+13x^5)(10m^4n-13x^5)$.

10. $(a-b+c)(a-b-c)$. 13. $(1+a-b)(1-a+b)$.

11. $(x^2+x+1)(x^2+x-1)$. 14. $(a^2+3a+1)(a^2-3a+1)$.

12. $(x+y+z)(x-y-z)$. 15. $(x+y+3)(x-y-3)$.

16. $(x^2+xy+y^2)(x^2-xy+y^2)$.

17. $(a^2+5a-4)(a^2-5a+4)$.

18. $(4x^2+3x-7)(4x^2-3x-7)$.

19. $(m^4+5m^2n^2+2n^4)(m^4-5m^2n^2-2n^4)$.

Product of Two Binomials having the Same First Term.

Let it be required to multiply $x+a$ by $x+b$.

$$\begin{array}{l}
x+a \\
x+b \\
\hline
x^2+ \quad ax \\
\quad\;\, + \quad bx+ab \\
\hline
\end{array}$$

Whence, $(x+a)(x+b)=x^2+(a+b)x+ab$.

That is, *the product of two binomials having the same first term equals the square of the first term, plus the algebraic sum of the second terms multiplied by the first term, plus the product of the second terms.*

1. Multiply $x-5$ by $x+3$.

The coefficient of x is the sum of -5 and $+3$, or -2.
The last term is the product of -5 and $+3$, or -15.

Whence, $(x-5)(x+3)=x^2-2x-15$.

2. Multiply $x-5$ by $x-3$.

The coefficient of x is the sum of -5 and -3, or -8.
The last term is the product of -5 and -3, or 15.

Whence, $(x-5)(x-3)=x^2-8x+15$.

3. Multiply $ab-4$ by $ab+7$.

The coefficient of ab is the sum of -4 and 7, or 3.
The last term is the product of -4 and 7, or -28.

Whence, $$(ab-4)(ab+7)=a^2b^2+3\,ab-28.$$

4. Multiply $x^2+6\,y^3$ by $x^2+8\,y^3$.

The coefficient of x^2 is the sum of $6\,y^3$ and $8\,y^3$, or $14\,y^3$.
The last term is the product of $6\,y^3$ and $8\,y^3$, or $48\,y^6$.

Whence, $$(x^2+6\,y^3)(x^2+8\,y^3)=x^4+14\,x^2y^3+48\,y^6.$$

EXERCISE 28

Expand the following by inspection:

1. $(x+3)(x+4)$.

2. $(x-2)(x+5)$.

3. $(x-11)(x-1)$.

4. $(a-7)(a+2)$.

5. $(a^2+15)(a^2+1)$.

6. $(m^3-3)(m^3+8)$.

7. $(x^n-2)(x^n-6)$.

8. $(a^m+10)(a^m+2)$.

9. $(mn-7)(mn-3)$.

10. $(ab+1)(ab-3)$.

11. $(x-y+7)(x-y-6)$.

12. $(a+8\,x)(a+9\,x)$.

13. $(x-9\,y)(x-5\,y)$.

14. $(m^3+6\,n)(m^3-7\,n)$.

15. $(a+b+2)(a+b+13)$.

16. $(x^{2m}+10\,y^{3n})(x^{2m}-9\,y^{3n})$.

17. $(a^3-9\,b^4)(a^3+8\,b^4)$.

18. $(mn-14\,xy)(mn-4\,xy)$.

19. $(m-n-3)(m-n+11)$.

20. $(a^2b+11\,c^3)(a^2b-12\,c^3)$.

100. Product of Two Binomials of the Form $mx+n$ and $px+q$. We find by multiplication:

$$\begin{array}{rl}
 & mx+n \\
 & \times \\
 & px+q \\
\hline
mpx^2+ & npx \\
+ & mqx+nq \\
\hline
mpx^2+ & (np+mq)\,x+nq.
\end{array}$$

he first term of this result, mpx^2, is the product of the first ıs of the binomial factors, and the last term, nq, the product he second terms.

he middle term, $(np+mq)x$, is the sum of the products of terms, in the binomial factors, connected by cross lines.

Ex. Multiply $3x+4$ by $2x-5$.

The first term is the product of $3x$ and $2x$, or $6x^2$.

The coefficient of x is the sum of 4×2 and $3 \times (-5)$; that is, $8-15$, -7.

The last term is the product of 4 and -5, or -20.

Whence, $(3x+4)(2x-5)=6x^2-7x-20$.

EXERCISE 29

Expand the following by inspection:

1. $(x+6)(3x+2)$. work
2. $(2x+1)(7x-1)$. HW
3. $(2x-5)(4x+3)$. [illegible]
4. $(4a-3)(5a-3)$.
5. $(4m+1)(4m+3)$.
6. $(3n+2)(5n-2)$.
7. $(2a^2-1)(11a^2-4)$
8. $(5x^4+6)(6x^4+1)$.
9. $(2ax-3)(5ax+6)$.
10. $(3x+2n)(10x-n)$.
11. $(4x-3y)(9x+2y)$.
12. $(7a-2m)(7a-4m)$.
13. $(6x^n+y)(9x^n+y)$.
14. $(6a^2+x^2)(8a^2-5x^2)$.
15. $(5m^3-2n^4)(10m^3-7n^4)$.
16. $(8ax-3by)(9ax+5by)$.
17. $[6(m+n)-5][(m+n)-2]$.
18. $[3(a-b)+4][4(a-b)-3]$.

101. We find by division,

$$\frac{a^2-b^2}{a+b}=a-b. \qquad \frac{a^2-b^2}{a-b}=a+b.$$

That is,

If the difference of the squares of two numbers be divided by the sum of the numbers, the quotient is the difference of the numbers.

If the difference of the squares of two numbers be divided by the difference of the numbers, the quotient is the sum of the numbers.

1. Divide $25y^2z^4-9$ by $5yz^2-3$.

By § 96, $25y^2z^4$ is the square of $5yz^2$; then, by the second rule,

$$\frac{25y^2z^4-9}{5yz^2-3}=5yz^2+3.$$

2. Divide $x^2-(y-z)^2$ by $x+(y-z)$.

By the first rule, $\dfrac{x^2-(y-z)^2}{x+(y-z)}=x-(y-z)=x-y+z.$

EXERCISE 30

Find, without actual division, the values of the following:

1. $\dfrac{a^2-4}{a+2}$.

2. $\dfrac{x^2-9}{x-3}$.

3. $\dfrac{25n^4-1}{5n^2-1}$.

4. $\dfrac{16x^6-81}{4x^3+9}$.

5. $\dfrac{36a^{2p}-121b^{4q}}{6a^p-11b^{2q}}$.

6. $\dfrac{64n^8-x^{10}}{8n^4+x^5}$.

7. $\dfrac{1-144a^{2n}b^{6m}}{1-12a^nb^{3m}}$.

8. $\dfrac{49x^6y^{10}-169z^8}{7x^3y^5+13z^4}$.

9. $\dfrac{225a^{12}-100b^{18}}{15a^6+10b^9}$.

10. $\dfrac{196m^2n^{14}-256x^{16}}{14mn^7-16x^8}$.

11. $\dfrac{4a^2-(b-c)^2}{2a-(b-c)}$.

12. $\dfrac{a^2-(m+3n)^2}{a+(m+3n)}$.

13. $\dfrac{(a+x)^2-(b-y)^2}{(a+x)+(b-y)}$.

102. We find by division,

$$\frac{a^3+b^3}{a+b}=a^2-ab+b^2. \qquad \frac{a^3-b^3}{a-b}=a^2+ab+b^2.$$

That is,

If the sum of the cubes of two numbers be divided by the sum of the numbers, the quotient is the square of the first ~~number~~, minus the product of the first ~~by the second~~, plus the square of the second ~~number~~.

If the difference of the cubes of two numbers be divided by the difference of the numbers, the quotient is the square of the first number, plus the product of the first by the second, plus the square of the second number.

1. Divide $1 + 8a^3$ by $1 + 2a$.

By § 96, $8a^3$ is the cube of $2a$; then, by the first rule,

$$\frac{1+8a^3}{1+2a} = \frac{1+(2a)^3}{1+2a} = 1 - 2a + (2a)^2 = 1 - 2a + 4a^2.$$

2. Divide $27x^6 - 64y^9$ by $3x^2 - 4y^3$.

By the second rule,

$$\frac{27x^6 - 64y^9}{3x^2 - 4y^3} = \frac{(3x^2)^3 - (4y^3)^3}{3x^2 - 4y^3} = (3x^2)^2 + (3x^2)(4y^3) + (4y^3)^2$$

$$= 9x^4 + 12x^2y^3 + 16y^6.$$

EXERCISE 31

Find, without actual division, the values of the following:

1. $\dfrac{x^3+1}{x+1}$.
2. $\dfrac{1-a^3}{1-a}$.
3. $\dfrac{n^3-27}{n-3}$.
4. $\dfrac{8+m^{3p}}{2+m^p}$.
5. $\dfrac{x^6y^{12}-z^9}{x^2y^4-z^3}$.
6. $\dfrac{a^6+b^6}{a^2+b^2}$.
7. $\dfrac{a^3+125}{a+5}$.
8. $\dfrac{64x^{6m}-1}{4x^{2m}-1}$.
9. $\dfrac{a^3b^3-216}{ab-6}$.
10. $\dfrac{343m^6+n^3}{7m^2+n}$.
11. $\dfrac{27x^6-125y^3}{3x^2-5y}$.
12. $\dfrac{343m^3n^3+8p^3}{7mn+2p}$.
13. $\dfrac{64a^6b^3+216c^9}{4a^2b+6c^3}$.
14. $\dfrac{x^{18}-1000y^{12}z^{15}}{x^6-10y^4z^5}$.
15. $\dfrac{729a^9x^3+512y^6}{9a^3x+8y^2}$.

103. We find by a[illegible]al division,

$$\frac{a^4 \ [illegible]}{a \ [illegible]} = a^3 - a^2b + ab^2 - b^3.$$

$$\frac{[illegible] - b^4}{a-b} = a^3 + a^2b + ab^2 + b^3.$$

$$\frac{a^5+b^5}{a+b}=a^4-a^3b+a^2b^2-ab^3+b^4.$$

$$\frac{a^5-b^5}{a-b}=a^4+a^3b+a^2b^2+ab^3+b^4;\ \text{etc.}$$

In these results, we observe the following laws:

I. *The exponent of a in the first term of the quotient is less by 1 than its exponent in the dividend, and decreases by 1 in each succeeding term.*

II. *The exponent of b in the second term of the quotient is 1, and increases by 1 in each succeeding term.*

III. *If the divisor is $a-b$, all the terms of the quotient are positive; if the divisor is $a+b$, the terms of the quotient are alternately positive and negative.*

A general proof of these laws will be found in § 466.

1. Divide a^7-b^7 by $a-b$.

By the above laws,

$$\frac{a^7-b^7}{a-b}=a^6+a^5b+a^4b^2+a^3b^3+a^2b^4+ab^5+b^6.$$

2. Divide $16x^4-81$ by $2x+3$.

We have, $$\frac{16x^4-81}{2x+3}=\frac{(2x)^4-3^4}{2x+3}$$

$$=(2x)^3-(2x)^2\cdot 3+2x\cdot 3^2-3^3$$

$$=8x^3-12x^2+18x-27.$$

EXERCISE 32

Find, without actual division, the values of the following:

1. $\dfrac{a^4-b^4}{a-b}$.

2. $\dfrac{m^5+n^5}{m+n}$.

3. $\dfrac{x^6-1}{x-1}$.

4. $\dfrac{1-a^6}{1+a}$.

5. $\dfrac{x^8-y^8}{x^2+y^2}$.

6. $\dfrac{a^{15}-b^5c^{10}}{a^3-bc^2}$.

7. $\dfrac{16a^{12}-x^8}{2a^3+x^2}$.	11. $\dfrac{a^8-b^8}{a-b}$.	15. $\dfrac{m^{6p}-729n^6}{m^p+3n}$.
8. $\dfrac{m^{4n}-81}{m^n-3}$.	12. $\dfrac{32x^5-1}{2x-1}$.	16. $\dfrac{81x^{12}-16y^8}{3x^3-2y^2}$.
9. $\dfrac{a^7-x^7}{a-x}$.	13. $\dfrac{256x^4-y^4}{4x+y}$.	17. $\dfrac{a^7+128b^{14}}{a+2b^2}$.
10. $\dfrac{n^7+1}{n+1}$.	14. $\dfrac{a^5+243x^{10m}}{a+3x^{2m}}$.	18. $\dfrac{64x^6-729}{2x-3}$.

104. The following statements will be found to be true if n is any positive integer:

I. *$a^n - b^n$ is always divisible by $a-b$.*

Thus, a^2-b^2, a^3-b^3, a^4-b^4, etc., are divisible by $a-b$.

II. *a^n-b^n is divisible by $a+b$ if n is even.*

Thus, a^2-b^2, a^4-b^4, a^6-b^6, etc., are divisible by $a+b$.

III. *a^n+b^n is divisible by $a+b$ if n is odd.*

Thus, a^3+b^3, a^5+b^5, a^7+b^7, etc., are divisible by $a+b$.

IV. *a^n+b^n is divisible by neither $a+b$ nor $a-b$ if n is even.*

Thus, a^2+b^2, a^4+b^4, a^6+b^6, etc., are divisible by neither $a+b$ nor $a-b$.

Proofs of the above statements will be found in § 467.

Take out the Common Monomial Number

VIII. FACTORING

105. To **Factor** an algebraic expression is to find two or more expressions which, when multiplied together, shall produce the given expression.

In the present chapter we consider only the separation of rational and integral expressions (§ 63), with integral numerical coefficients, into factors of the same form.

A **Common Factor** of two or more expressions is an expression which will exactly divide each of them.

106. It is not always possible to factor an expression; there are, however, certain forms which can always be factored; these will be considered in the present treatise.

107. Case I. *When the terms of the expression have a common factor.*

1. Factor $14\,ab^4 - 35\,a^3b^2$.

Each term contains the monomial factor $7\,ab^2$.
Dividing the expression by $7\,ab^2$, we have $2\,b^2 - 5\,a^2$.

Then, $$14\,ab^4 - 35\,a^3b^2 = 7\,ab^2\,(2\,b^2 - 5\,a^2).$$

2. Factor $(2\,m + 3)\,x^2 + (2\,m + 3)\,y^3$.

The terms have the common binomial factor $2\,m + 3$.
Dividing the expression by $2\,m + 3$, we have $x^2 + y^3$.

Then, $$(2\,m + 3)\,x^2 + (2\,m + 3)\,y^3 = (2\,m + 3)\,(x^2 + y^3).$$

3. Factor $(a - b)\,m + (b - a)\,n$.

By § 52, $$b - a = -(a - b).$$

Then, $$(a - b)\,m + (b - a)\,n = (a - b)\,m - (a - b)\,n$$
$$= (a - b)\,(m - n).$$

We may also solve Ex. 3 as follows:

$$(a - b)\,m + (b - a)\,n = (b - a)\,n - (b - a)\,m = (b - a)\,(n - m).$$

4. Factor $5a(x-y)-3a(x+y)$.

$$5a(x-y)-3a(x+y)=a[5(x-y)-3(x+y)]$$
$$=a(5x-5y-3x-3y)$$
$$=a(2x-8y)=2a(x-4y).$$

EXERCISE 33

Factor the following:

1. $63x^9-54x^4$.
2. $a^6-5a^5-2a^4+3a^3$.
3. $m^5n^2+m^3n^4-mn^6$.
4. $24x^3y^3-40x^2y^2+56x^4y$.
5. $(a-2)b^4-(a-2)c^3d^5$.
6. $(3x+5)m+(3x+5)$.
7. $(m-n)(x+y)-(n-m)z$.
8. $a(a^2-2)+3(2-a^2)$.
9. $(x+y)(m+n)+(x+y)(m-n)$.
10. $a(b+c)-a(b-c)$.
11. $3x^2(x-1)-(1-x)$.
12. $6(3a+4b)+6(5a-2b)$.
13. $5(2x-y)-5(x+3y)$.
14. $(a+m)^2-3(a+m)$.
15. $x^2(5y-2z)-x^2(2y+z)$.
16. $(m-n)^3+2m(m-n)^2$.
17. $3a^{2n+5}-7a^{n+2}b+a^n$.
18. $(a-b)(m^2+xz)-(a-b)(m^2-yz)$.
19. $(m-n)^4-2m(m-n)^3+m^2(m-n)^2$.

108. The terms of a polynomial may sometimes be so arranged as to show a common binomial factor; and the expression can then be factored as in § 107.

1. Factor $ab-ay+bx-xy$.

By § 107, $ab-ay+bx-xy=a(b-y)+x(b-y)$.
The terms now have the common factor $b-y$.

Whence, $ab-ay+bx-xy=(b-y)(a+x)$.

2. Factor a^3+2a^2-3a-6.

If the third term is negative, it is convenient to enclose the last two terms in parentheses preceded by a $-$ sign.

Thus, $a^3 + 2a^2 - 3a - 6 = (a^3 + 2a^2) - (3a + 6)$

$= a^2(a+2) - 3(a+2) = (a+2)(a^2-3).$

EXERCISE 34

Factor the following:

1. $ac + ad + bc + bd.$
2. $xy - 3x + 2y - 6.$
3. $mx + my - nx - ny.$
4. $ab - a - 5b + 5.$
5. $8xy + 12ay + 10bx + 15ab.$
6. $m^4 + 6m^3 - 7m - 42.$
7. $6 - 10a + 27a^2 - 45a^3.$
8. $20ab - 28ad - 5bc + 7cd.$
9. $m^3 - m^2n + mn^2 - n^3.$
10. $a^5b^5 - a^3b^2c^2d^3 - a^2b^3c^3d^2 + c^5d^5.$
11. $63 + 36x^2 + 56x^3 + 32x^5.$
12. $48xy + 18nx - 88my - 33mn.$
13. $mx + my + nx + ny + px + py.$
14. $ax - ay + az - bx + by - bz.$
15. $3am - 6an + 4bm - 8bn + cm - 2cn.$
16. $ax + ay - az - bx - by + bz + cx + cy - cz.$

109. If an expression can be resolved into two equal factors, it is said to be a *perfect square*, and one of the equal factors is called its *square root.*

Thus, since $9a^4b^2$ is equal to $3a^2b \times 3a^2b$, it is a perfect square, and $3a^2b$ is its square root.

$9a^4b^2$ is also equal to $(-3a^2b) \times (-3a^2b)$; so that $-3a^2b$ is also its square root; in the examples of the present chapter, we shall consider the *positive* square root only.

110. The following rule for extracting the positive square root of a monomial perfect square is evident from § 109:

Extract the square root of the numerical coefficient, and divide the exponent of each letter by 2.

Thus, the square root of $25a^4b^6c^2$ is $5a^2b^3c$.

111. It follows from § 97 that a trinomial is a perfect square when its first and last terms are perfect squares and positive, and the second term plus or minus twice the product of their square roots.

Thus, in the expression $4x^2 - 12xy + 9y^2$, the square root of the first term is $2x$, and of the last term $3y$; and the second term is equal to $-2(2x)(3y)$.

Whence, $4x^2 - 12xy + 9y^2$ is a perfect square.

112. To find the square root of a trinomial perfect square, we reverse the rule of § 97:

Extract the square roots (§ 110) *of the first and third terms, and connect the results by the sign of the second term.*

1. Find the square root of $4x^2 + 12xy + 9y^2$.

By the rule, the result is $2x + 3y$.

(The expression may be written in the form

$$(-2x)^2 + 2(-2x)(-3y) + (-3y)^2,$$

which shows that $(-2x) + (-3y)$, or $-2x - 3y$, is also its square root; but the first form is simpler, and will be used in all the examples of the present chapter.)

2. Find the square root of $m^2 - 2mn + n^2$.

By the rule, the result is $m - n$.

(The expression may also be written $n^2 - 2mn + m^2$; in which case, by the rule, its square root is $n - m$.)

113. Case II. *When the expression is a trinomial perfect square.*

1. Factor $25a^2 + 10ab^2 + b^4$.

By § 112, the square root of the expression is $5a + b^2$.
Then, $25a^2 + 10ab^2 + b^4 = (5a + b^2)^2$.

2. Factor $m^4 - 4m^2n^2 + 4n^4$.

By § 112, the square root of the expression is either $m^2 - 2n^2$, or $2n^2 - m^2$.

Then, $m^4 - 4m^2n^2 + 4n^4 = (m^2 - 2n^2)^2$, or $(2n^2 - m^2)^2$.

3. Factor $x^2 - 2x(y-z) + (y-z)^2$.

We have $\quad x^2 - 2x(y-z) + (y-z)^2$

$$= [x - (y-z)]^2 = (x - y + z)^2;$$

or, $$= [(y-z) - x]^2 = (y - z - x)^2.$$

4. Factor $-9a^4 - 6a^2 - 1$.

$$-9a^4 - 6a^2 - 1 = -(9a^4 + 6a^2 + 1) = -(3a^2 + 1)^2.$$

EXERCISE 35

Factor the following:

1. $x^2 + 8x + 16$.
2. $9 - 6a + a^2$.
3. $m^2 + 10mn + 25n^2$.
4. $4a^6 - 4a^3bc^2 + b^2c^4$.
5. $x^2y^2 + 14xy + 49$.
6. $36a^2 - 132ab + 121b^2$.
7. $-16a^2 + 24ax - 9x^2$.
8. $81m^2 + 180mn + 100n^2$.
9. $-25x^{10} - 60x^5y^3z^2 - 36y^6z^4$.
10. $64a^2x^2 - 240abxy + 225b^2y^2$.
11. $49m^{2n} + 168m^nx^p + 144x^{2p}$.
12. $100a^2b^2 + 180abc^2 + 81c^4$.
13. $144x^{4n}y^2 - 312x^{2n}yz^{3m} + 169z^{6m}$.
14. $-121a^4m^2 + 220a^2b^2mn - 100b^4\cancel{n^2}$.
15. $169a^8b^2 + 364a^4bc^2d^3 + 196c^4d^6$.
16. $(x+y)^2 + 22(x+y) + 121$.
17. $a^2 - 8a(m-n) + 16(m-n)^2$.
18. $9x^2 - 6x(y+z) + (y+z)^2$.
19. $(m-n)^2 - 2(m-n)n + n^2$.
20. $25(a+b)^2 + 40(a+b)c + 16c^2$.

21. $36(a-x)^2-84(a-x)y+49y^2.$

22. $49m^2+42m(m+x)+9(m+x)^2.$

23. $(a+b)^2+4(a+b)(a-b)+4(a-b)^2.$

24. $9(x+y)^2-12(x+y)(x-y)+4(x-y)^2.$

114. Case III. *When the expression is the difference of two perfect squares.*

By § 98, $$a^2-b^2=(a+b)(a-b).$$

Hence, to obtain the factors, we reverse the rule of § 98:

Extract the square root of the first square, and of the second square; add the results for one factor, and subtract the second result from the first for the other.

1. Factor $36a^2b^4-49c^6$.

The square root of $36a^2b^4$ is $6ab^2$, and of $49c^6$ is $7c^3$.

Then, $$36a^2b^4-49c^6=(6ab^2+7c^3)(6ab^2-7c^3).$$

2. Factor $(2x-3y)^2-(x-y)^2$.

By the rule, $$\begin{aligned}(2x-3y)^2-(x-y)^2 &= [(2x-3y)+(x-y)][(2x-3y)-(x-y)]\\ &=(2x-3y+x-y)(2x-3y-x+y)\\ &=(3x-4y)(x-2y).\end{aligned}$$

A polynomial of more than two terms may sometimes be expressed as the difference of two perfect squares, and factored by the rule of Case III.

3. Factor $2mn+m^2-1+n^2$.

The first, second, and last terms may be grouped together in the order $m^2+2mn+n^2$; which expression, by § 112, is the square of $m+n$.

Thus, $$\begin{aligned}2mn+m^2-1+n^2 &= (m^2+2mn+n^2)-1\\ &=(m+n)^2-1\\ &=(m+n+1)(m+n-1).\end{aligned}$$

4. Factor $12y+x^2-9y^2-4$.

$$
\begin{aligned}
12y + x^2 - 9y^2 - 4 &= x^2 - 9y^2 + 12y - 4 \\
&= x^2 - (9y^2 - 12y + 4) \\
&= x^2 - (3y - 2)^2, \text{ by § 112,} \\
&= [x + (3y - 2)][x - (3y - 2)] \\
&= (x + 3y - 2)(x - 3y + 2).
\end{aligned}
$$

5. Factor $a^2 - c^2 + b^2 - d^2 - 2cd - 2ab$.

$$
\begin{aligned}
&a^2 - c^2 + b^2 - d^2 - 2cd - 2ab \\
&\quad = a^2 - 2ab + b^2 - c^2 - 2cd - d^2 \\
&\quad = a^2 - 2ab + b^2 - (c^2 + 2cd + d^2) = (a - b)^2 - (c + d)^2, \text{ by § 112,} \\
&\quad = [(a - b) + (c + d)][(a - b) - (c + d)] \\
&\quad = (a - b + c + d)(a - b - c - d).
\end{aligned}
$$

EXERCISE 36

Factor the following:

1. $x^6 - y^4$.

2. $4a^2 - 1$.

3. $n^6 - 9$.

4. $16 - 25a^8$.

5. $1 - 64m^2n^2$.

6. $36x^4 - 121y^6$.

7. $1 - 81a^6b^2c^4$.

8. $49a^8 - 144b^4x^{2p}$.

9. $100m^4x^{12} - 169n^6y^{10}$.

10. $225x^8y^2 - 196z^4$.

11. $324a^{4m}b^{10n} - 625$.

12. $361x^{14} - 256y^{20}c^{16}$.

13. $(a - b)^2 - c^2$.

14. $(5x + y)^2 - x^2$.

15. $a^2 - (m + n)^2$.

16. $a^2 - (b - 2c)^2$.

17. $(x + 4y)^2 - 9z^2$.

18. $36m^2 - (2m - 3)^2$.

19. $(x - y)^2 - (m + n)^2$.

20. $(2a + x)^2 - (b + 3y)^2$.

21. $(a - b)^2 - (c - d)^2$.

22. $(2m + n)^2 - (m + 2n)^2$.

23. $(6a + x)^2 - (a - 8x)^2$.

24. $(9x - 5y)^2 - (6x - 7y)^2$.

25. $25(8a - 3b)^2 - 9(4a + 5b)^2$

26. $(a + b - c)^2 - (a - b + c)^2$.

27. $(m + n + 3)^2 - (m - n - 4)^2$.

28. $a^2 + 2ab + b^2 - c^2$.

29. $x^2 - y^2 - 2yz - z^2$.

30. $m^2 - n^2 + 2np - p^2$.

31. $a^2 + b^2 - 1 - 2\,ab.$

32. $y^2 + 2\,xy - 4 + x^2.$

33. $4\,m^2 - 4\,mn + n^2 - p^2.$

34. $9\,a^2 + 16\,b^2 - 25\,c^2 + 24\,ab.$

35. $9 - a^2 + 2\,ab - b^2.$

36. $4\,m^2 - p^2 - 9\,n^2 - 6\,np.$

37. $12\,yz + 16\,x^2 - 9\,z^2 - 4\,y^2.$

38. $m^2 - 2\,mn + n^2 - x^2 + 2\,xy - y^2.$

39. $a^2 + 2\,ab + b^2 - c^2 - 2\,cd - d^2.$

40. $a^2 + x^2 - b^2 - y^2 + 2\,ax + 2\,by.$

41. $x^2 - y^2 + m^2 - 1 - 2\,mx - 2\,y.$

42. $a^2 - 4\,ax + 4\,x^2 - b^2 + 6\,by - 9\,y^2.$

43. $16\,a^2 - 8\,ab + b^2 - c^2 - 10\,cd - 25\,d^2.$

44. $28\,xy - 36\,z^2 + 49\,y^2 + 60\,z - 25 + 4\,x^2.$

115. Case IV. *When the expression is in the form*

$$x^4 + ax^2y^2 + y^4.$$

Certain trinomials of the above form may be factored by expressing them as the difference of two perfect squares, and then employing § 114.

1. Factor $a^4 + a^2b^2 + b^4.$

By § 111, a trinomial is a perfect square if its first and last terms are perfect squares and positive, and its second term plus or minus twice the product of their square roots.

The given expression can be made a perfect square by adding a^2b^2 to its second term; and this can be done provided we subtract a^2b^2 from the result.

Thus, $a^4 + a^2b^2 + b^4 = (a^4 + 2\,a^2b^2 + b^4) - a^2b^2$

$= (a^2 + b^2)^2 - a^2b^2$, by § 112,

$= (a^2 + b^2 + ab)(a^2 + b^2 - ab)$, by § 114,

$= (a^2 + ab + b^2)(a^2 - ab + b^2).$

2. Factor $9\,x^4 - 37\,x^2 + 4.$

The expression will be a perfect square if its second term is $-12\,x^2.$

Thus, $9x^4 - 37x^2 + 4 = (9x^4 - 12x^2 + 4) - 25x^2$

$$= (3x^2 - 2)^2 - (5x)^2$$

$$= (3x^2 + 5x - 2)(3x^2 - 5x - 2).$$

(The expression may also be factored as follows:

$$9x^4 - 37x^2 + 4 = (9x^4 + 12x^2 + 4) - 49x^2$$

$$= (3x^2 + 2)^2 - (7x)^2 = (3x^2 + 7x + 2)(3x^2 - 7x + 2)$$

Several expressions in Exercise 37 may be factored in two different ways.

The factoring of trinomials of the form $x^4 + ax^2y^2 + y^4$, when the factors involve surds, will be considered in § 300.)

EXERCISE 37

Factor the following:

1. $x^4 + 5x^2 + 9$.
2. $a^4 - 21a^2b^2 + 36b^4$.
3. $4 - 33x^2 + 4x^4$.
4. $25m^4 - 14m^2n^2 + n^4$.
5. $9x^4 + 6x^2y^2 + 49y^4$.
6. $16a^4 - 81a^2 + 16$.
7. $64 - 64m^2 + 25m^4$.
8. $49a^4 - 127a^2x^2 + 81x^4$.

Factor each of the following in two different ways (compare Ex. 2, § 115):

9. $x^4 - 17x^2 + 16$.
10. $9 - 148a^2 + 64a^4$.
11. $16m^4 - 104m^2x^2 + 25x^4$.
12. $36a^4 - 97a^2m^2 + 36m^4$.

116. Case V. *When the expression is in the form*

$$x^2 + ax + b.$$

We saw, in § 99, that the product of two binomials of the form $x + m$ and $x + n$, was in the form $x^2 + ax + b$; where the coefficient of x was the algebraic sum of the second terms of the binomials, and the third term the product of the second terms of the binomials.

In certain cases, it is possible to reverse the process, and resolve a trinomial of the form $x^2 + ax + b$ into two binomial factors of the form $x + m$ and $x + n$.

To obtain the second terms of the binomials, we simply reverse the rule of § 99, and *find two numbers whose algebraic sum is the coefficient of x, and whose product is the last term of the trinomial.*

The numbers may be found by inspection.

1. Factor $x^2 + 14x + 45$.

We find two numbers whose sum is 14 and product 45.

By inspection, we determine that these numbers are 9 and 5.

Whence, $$x^2 + 14x + 45 = (x + 9)(x + 5).$$

2. Factor $x^2 - 5x + 4$.

We find two numbers whose sum is -5 and product 4.

Since the sum is negative, and the product positive, the numbers must both be negative.

By inspection, we determine that the numbers are -4 and -1.

Whence, $$x^2 - 5x + 4 = (x - 4)(x - 1).$$

3. Factor $x^6 + 6x^3 - 16$.

We find two numbers whose sum is 6 and product -16.

Since the sum is positive, and the product negative, the numbers must be of opposite sign; and the positive number must have the greater absolute value.

By inspection, we determine that the numbers are $+8$ and -2.

Whence, $$x^6 + 6x^3 - 16 = (x^3 + 8)(x^3 - 2).$$

4. Factor $x^4 - abx^2 - 42a^2b^2$.

We find two numbers whose sum is -1 and product -42.

The numbers must be of opposite sign, and the negative number must have the greater absolute value.

By inspection, we determine that the numbers are -7 and $+6$.

Whence, $$x^4 - abx^2 - 42a^2b^2 = (x^2 - 7ab)(x^2 + 6ab).$$

5. Factor $1 + 2a - 99a^2$.

We find two numbers whose sum is $+2$ and product -99.

By inspection, we determine that the numbers are $+11$ and -9.

Whence, $$1 + 2a - 99a^2 = (1 + 11a)(1 - 9a).$$

If the x^2 term is negative, the entire expression should be enclosed in parentheses preceded by a $-$ sign.

6. Factor $24 + 5x - x^2$.

We have, $$24 + 5x - x^2 = -(x^2 - 5x - 24)$$
$$= -(x - 8)(x + 3) = (8 - x)(3 + x);$$

changing the sign of each term of the first factor.

(In case the numbers are large, we may proceed as follows:

Required the numbers whose sum is -26 and product -192.

One of the numbers must be $+$, and the other $-$.

Taking in order, beginning with the factors $+1 \times -192$, all possible pairs of factors of -192, of which one is $+$ and the other $-$, we have:

$$+1 \times -192.$$
$$+2 \times -96.$$
$$+3 \times -64.$$
$$+4 \times -48.$$
$$+6 \times -32.$$

Since the sum of $+6$ and -32 is -26, they are the numbers required.)

EXERCISE 38

Factor the following:

1. $x^2 + 4x + 3$.
2. $x^2 - 7x + 10$.
3. $a^2 + 7a - 18$.
4. $m^2 - 14m - 15$.
5. $y^2 - 16y + 55$.
6. $x^2 + 16x + 39$.
7. $28 + 3c - c^2$.
8. $66 - 5n - n^2$.
9. $a^2 - 14a + 48$.
10. $x^2 + 20x + 51$.
11. $x^2 - 12x - 45$.
12. $n^2 + 14n - 32$.
13. $x^2 - 17x + 52$.
14. $a^2 + 18a + 56$.
15. $84 + 5x - x^2$.
16. $y^2 + 16y - 57$.
17. $x^2 - 10x - 75$.
18. $m^2 + 19m + 90$.
19. $95 - 14n - n^2$.
20. $x^2 - 20x + 96$.
21. $a^2 + 21a + 98$.
22. $x^2 - 7x - 78$.
23. $105 - 8m - m^2$.
24. $c^4 - 21c^2 + 104$.

25. $x^4 - 23x^2 + 76$.

26. $a^6 + a^3 - 110$.

27. $n^{10} - 16n^5 - 80$.

28. $a^{2n} + 18a^n + 65$.

29. $x^{2m} + 11x^m - 12$.

30. $c^{4p} - 19c^{2p} + 88$.

31. $x^2y^6 - 13xy^3 - 30$.

32. $a^2b^4 - 23ab^2 + 112$.

33. $n^2x^2 + 25nx + 154$.

34. $126 + 15y^4 - y^8$.

35. $a^4x^4 + 9a^2x^2 - 162$.

36. $m^{6n} - 23m^{3n} + 120$.

37. $(a+b)^2 + 14(a+b) + 24$.

38. $(x-y)^2 - 15(x-y) - 16$.

39. $(m-n)^2 + 21(m-n) - 130$.

40. $(a+x)^2 - 28(a+x) + 192$.

41. $a^2 + 6ax + 5x^2$.

42. $x^2 - 7xy - 8y^2$.

43. $1 + 5a - 14a^2$.

44. $m^2 - 17mn + 66n^2$.

45. $a^2 + 12ab + 27b^2$.

46. $x^2 - 14mx + 40m^2$.

47. $1 - 9x - 36x^2$.

48. $m^2 + 3mn - 54n^2$.

49. $x^2 + 12xy + 20y^2$.

50. $a^2b^2 - 17abc + 60c^2$.

51. $1 - 13n - 68n^2$.

52. $a^2 + 15ax - 100x^2$.

53. $1 + 17mn + 70m^2n^2$.

54. $x^6 - 17x^3yz^2 + 72y^2z^4$.

55. $a^{8m} + 6a^{4m}b - 91b^2$.

56. $1 - 3xy - 108x^2y^2$.

57. $a^2 - 32abc + 112b^2c^2$.

58. $x^4y^4 + 29x^2y^2z - 170z^2$.

59. $x^2 - (2m+3n)x + 6mn$.

60. $x^2 - (a-b)x - ab$.

117. Case VI. *When the expression is in the form*

$$ax^2 + bx + c.$$

We saw, in § 100, that the product of two binomials of the form $mx + n$ and $px + q$, was in the form $ax^2 + bx + c$; where the first term was the product of the first terms of the binomial factors, and the last term the product of the second terms.

Also, the middle term was the sum of the products of the terms, in the binomial factors, connected by cross lines.

In certain cases it is possible to resolve a trinomial of the form $ax^2 + bx + c$ into two binomial factors of the form $mx + n$ and $px + q$.

1. Factor $3x^2 + 8x + 4$.

The first terms of the binomial factors must be such that their product is $3x^2$; they can be only $3x$ and x.

The second terms must be such that their product is 4.

The numbers whose product is 4 are 4 and 1, -4 and -1, 2 and 2, and -2 and -2; the possible cases are represented below:

$$\begin{array}{ccc} x+4 & x+1 & x-4 \\ 3x+1 & 3x+4 & 3x-1 \\ \hline 13x & 7x & -13x \end{array}$$

$$\begin{array}{ccc} x-1 & x+2 & x-2 \\ 3x-4 & 3x+2 & 3x-2 \\ \hline -7x & 8x & -8x \end{array}$$

The corresponding middle term of the trinomial, obtained by cross-multiplication, as in § 100, is given in each case; and only the factors $x + 2$, $3x + 2$ satisfy the condition that the middle term shall be $8x$.

Then, $$3x^2 + 8x + 4 = (x + 2)(3x + 2).$$

2. Factor $6x^2 - x - 2$.

The first terms of the factors must be $6x$ and x, or $3x$ and $2x$.

The second terms must be 2 and -1, or -2 and 1.

The possible cases are given below:

$$\begin{array}{cccc} 6x+2 & 6x-1 & 6x-2 & 6x+1 \\ x-1 & x+2 & x+1 & x-2 \\ \hline -4x & 11x & 4x & -11x \end{array}$$

$$\begin{array}{cccc} 3x+2 & 3x-1 & 3x-2 & 3x+1 \\ 2x-1 & 2x+2 & 2x+1 & 2x-2 \\ \hline x & 4x & -x & -4x \end{array}$$

Only the factors $3x - 2$ and $2x + 1$ satisfy the condition that the middle term shall be $-x$.

Then, $$6x^2 - x - 2 = (3x - 2)(2x + 1).$$

The following suggestions will be found of service:

(*a*) *If the last term of the trinomial is positive, the last terms of the factors will be both* $+$, *or both* $-$, *according as the middle term of the trinomial is* $+$ *or* $-$.

Thus, in Ex. 1, we need not have tried the numbers -1 and -4, nor -2 and -2; this would have left only three cases to consider.

(*b*) *If the last term of the trinomial is negative, the last terms of the factors will be one* $+$, *the other* $-$.

If the x^2 term is negative, the entire expression should be enclosed in parentheses preceded by a $-$ sign.

If the coefficient of x^2 is a perfect square, and the coefficient of x divisible by the square root of the coefficient of x^2, the expression may be readily factored by the method of § 116.

3. Factor $9x^2-18x+5$.

In this case, 18 is divisible by the square root of 9.

We have $\quad 9x^2-18x+5=(3x)^2-6(3x)+5.$

We find two numbers whose sum is -6, and product 5.
The numbers are -5 and -1.

Then, $\quad 9x^2-18x+5=(3x-5)(3x-1).$

EXERCISE 39

Factor the following:

1. $2x^2+9x+9$.
2. $3x^2-11x-20$.
3. $4x^2-28x+45$.
4. $6x^2+7x-3$.
5. $5x^2-36x+36$.
6. $16x^2+56x+33$.
7. $8n^2+18n-5$.
8. $4x^2-3x-7$.
9. $9x^2+12x-32$.
10. $6x^2+7ax+2a^2$.
11. $25x^2-25mx-6m^2$.
12. $10x^2-39x+14$.
13. $12x^2+11x+2$.
14. $20a^2x^2-23ax+6$.
15. $36x^2+12x-35$.
16. $6-x-15x^2$.
17. $5+9x-18x^2$.
18. $72+7x-49x^2$.
19. $24x^2-17nx+3n^2$.
20. $28x^2-x-2$.
21. $21x^{2m}+23x^my^{2n}+6y^{4n}$.
22. $18x^2-27abx-35a^2b^2$.
23. $24a^4+26a^2-5$.

118. It is not possible to factor every expression of the form $x^2 + ax + b$ by the method of § 116.

Thus, let it be required to factor $x^2 + 18x + 35$.

We must find two numbers whose sum is 18, and product 35.

The only pairs of positive integral factors of 35 are 7 and 5, and 35 and 1; and in neither case is the sum 18.

It is also impossible to factor every expression of the form $ax^2 + bx + c$ by the method of § 117.

Thus, it is impossible to find two binomial factors of the expression $4x^2 + 4x - 1$ by the method of § 117.

In § 298 will be given a *general* method for the factoring of any expression of the forms $x^2 + ax + b$, or $ax^2 + bx + c$.

119. If an expression can be resolved into three equal factors, it is said to be a *perfect cube*, and one of the equal factors is called its *cube root*.

Thus, since $27a^6b^3$ is equal to $3a^2b \times 3a^2b \times 3a^2b$, it is a perfect cube, and $3a^2b$ is its cube root.

120. The following rule for extracting the cube root of a positive monomial perfect cube is evident from § 119:

Extract the cube root of the numerical coefficient, and divide the exponent of each letter by 3.

Thus, the cube root of $125a^6b^9c^3$ is $5a^2b^3c$.

121. Case VII. *When the expression is the sum or difference of two perfect cubes.*

By § 102, the sum or difference of two perfect cubes is divisible by the sum or difference, respectively, of their cube roots; in either case the quotient may be obtained by the rules of § 102.

1. Factor $x^6 - 27y^9z^3$.

By § 120, the cube root of x^6 is x^2, and of $27y^9z^3$ is $3y^3z$.

Then one factor is $x^2 - 3y^3z$.

Dividing $x^6 - 27y^9z^3$ by $x^2 - 3y^3z$, the quotient is

$$x^4 + 3x^2y^3z + 9y^6z^2 \text{ (§ 102).}$$

Then, $x^6 - 27y^9z^3 = (x^2 - 3y^3z)(x^4 + 3x^2y^3z + 9y^6z^2)$.

2. Factor $a^6 + b^6$.

One factor is $a^2 + b^2$.
Dividing $a^6 + b^6$ by $a^2 + b^2$, the quotient is $a^4 - a^2b^2 + b^4$.
Then, $a^6 + b^6 = (a^2 + b^2)(a^4 - a^2b^2 + b^4)$.

3. Factor $(x + a)^3 - (x - a)^3$.

$$
\begin{aligned}
(x+a)^3 - (x-a)^3 \\
&= [(x+a) - (x-a)][(x+a)^2 + (x+a)(x-a) + (x-a)^2] \\
&= (x + a - x + a)(x^2 + 2\,ax + a^2 + x^2 - a^2 + x^2 - 2\,ax + a^2) \\
&= 2\,a\,(3\,x^2 + a^2).
\end{aligned}
$$

EXERCISE 40

Factor the following:

1. $a^3 + b^3$.
2. $x^3 - y^3$.
3. $1 + m^3n^3$.
4. $a^9 - b^3c^6$.
5. $8\,a^3 + 1$.
6. $1 - 27\,n^3$.
7. $a^6 + 1$.
8. $x^6y^6 + z^6$.
9. $64\,m^3 - n^3$.
10. $a^3b^3 - 216\,c^3$.
11. $8\,m^{3p} + 27\,n^{3q}$.
12. $27\,x^{6m} - 125\,y^{9n}$.
13. $64 + 125\,a^3b^3$.
14. $343\,a^3 - 64\,m^6$.
15. $125\,x^3 + 216\,y^3z^3$.
16. $27\,a^6 - 512\,b^9$.
17. $216\,a^3m^6 - 343\,n^9$.
18. $729\,a^3b^3 + 8\,c^3d^3$.
19. $(x + y)^3 + (x - y)^3$.
20. $m^3 - (m + n)^3$.
21. $27\,(a - b)^3 + 8\,b^3$.
22. $(2\,a + x)^3 - (a + 2\,x)^3$.
23. $(5\,x - 2\,y)^3 - (3\,x - 4\,y)^3$.

122. CASE VIII. *When the expression is the sum or difference of two equal odd powers of two numbers.*

By § 104, the sum or difference of two equal odd powers of two numbers is divisible by the sum or difference, respectively, of the numbers.

The quotient may be obtained by the laws of § 103.

Ex. Factor $a^5 + b^5$.

By § 104, one factor is $a + b$.

Dividing $a^5 + b^5$ by $a + b$, the quotient is

$$a^4 - a^3b + a^2b^2 - ab^3 + b^4. \qquad (\S\ 103)$$

Then, $\quad (a^5 + b^5) = (a + b)(a^4 - a^3b + a^2b^2 - ab^3 + b^4).$

EXERCISE 41

Factor the following:

1. $x^5 + y^5$.
2. $a^5 - 1$.
3. $1 - m^5n^5$.
4. $a^7 - b^7$.
5. $1 + x^7$.
6. $x^9 + n^9$.
7. $a^9 - 1$.
8. $n^{5p} + 32$.
9. $32a^5 - b^5$.
10. $243x^5 + y^5$.
11. $m^{14} + 128n^7$.
12. $32a^5b^{15n} - 243c^{10p}$.

123. By application of the rules already given, an expression may often be resolved into more than two factors.

If the terms of the expression have a common factor, the method of § 107 should always be applied first.

1. Factor $2ax^3y^2 - 8axy^4$.

By § 107, $\quad 2ax^3y^2 - 8axy^4 = 2axy^2(x^2 - 4y^2)$

$$= 2axy^2(x + 2y)(x - 2y), \text{ by } \S\ 114.$$

2. Factor $a^6 - b^6$.

By § 114, $\quad a^6 - b^6 = (a^3 + b^3)(a^3 - b^3).$

Whence, by § 121,

$$a^6 - b^6 = (a + b)(a^2 - ab + b^2)(a - b)(a^2 + ab + b^2).$$

3. Factor $x^8 - y^8$.

By § 114,

$$\begin{aligned} x^8 - y^8 &= (x^4 + y^4)(x^4 - y^4) \\ &= (x^4 + y^4)(x^2 + y^2)(x^2 - y^2) \\ &= (x^4 + y^4)(x^2 + y^2)(x + y)(x - y). \end{aligned}$$

4. Factor $3(m + n)^2 - 2(m^2 - n^2)$.

$$\begin{aligned} 3(m + n)^2 - 2(m^2 - n^2) &= 3(m + n)^2 - 2(m + n)(m - n) \\ &= (m + n)[3(m + n) - 2(m - n)] \\ &= (m + n)(3m + 3n - 2m + 2n) \\ &= (m + n)(m + 5n). \end{aligned}$$

5. Factor $a(a-1)-b(b-1)$.

$$\begin{aligned} a(a-1)-b(b-1) &= a^2-a-b^2+b \\ &= a^2-b^2-a+b \\ &= (a+b)(a-b)-(a-b) \\ &= (a-b)(a+b-1). \end{aligned}$$

The following is a list of the *type-forms* in factoring, considered in the present chapter:

$ax+ay-az$.	(§ 107)	x^2+ax+b.	(§ 116)
$a^2+2ab+b^2$.	(§ 113)	ax^2+bx+c.	(§ 117)
$a^2-2ab+b^2$.	(§ 113)	a^3+b^3.	(§ 121)
a^2-b^2.	(§ 114)	a^3-b^3.	(§ 121)
$x^4+ax^2y^2+y^4$.	(§ 115)	a^n-b^n.	(§ 122)

a^n+b^n, n odd. (§ 122)

MISCELLANEOUS AND REVIEW EXAMPLES

EXERCISE 42

Factor the following:

1. $x^2+21x+98$.
2. $a^2-20a-69$.
3. $12x^6-18x^5+6x^4-9x^3$.
4. $49a^4c^2-210a^2b^3c+225b^6$.
5. $21a^3+14a^2-24a-16$.
6. a^4-22a^2+120.
7. $49x^2+49x+12$.
8. $7x^2(3a-2b)-3x^2(2a-3b)$.
9. x^4-16.
10. $24ab-18ay-20bx+15xy$.
11. $27a^3+1000$.
12. $686m^4-250mn^3$.
13. $(x^2+x-2)^2-(x^2-x+3)^2$.
14. $64-n^6$.
15. $45n^3+18n^2-20n-8$.
16. $96-20a-a^2$.
17. $36x^4-69x^2+25$.
18. $2a^7x-8a^5x^3+2a^3x^5-8ax^7$.
19. $128x^7y^2+288x^4y^4+162xy^6$.
20. $2xy-2x^2y^2-264x^3y^3$.

21. $m^{16}-1$.

22. $a^{12}-1$.

23. $a^6b^2-30\,a^3bc^2+216\,c^4$.

24. $-121\,m^8+22\,m^4-1$.

25. $36\,x^6+24\,x^5-21\,x^4$.

26. $a^2b^3+a^2y^3-b^3x^2-x^2y^3$.

27. $\cancel{(a+2\,b)^2+8\,(a+2\,b)\,(2\,a-b)+16\,(2\,a-b)^2}$.

28. $4\,x\,(a-b-c)+5\,y\,(b+c-a)$.

29. $(m+n)^4-2\,(m+n)^3+(m+n)^2$.

30. $x^6-16\,x^3y^3+64\,y^6$.

31. $81\,m^4-256\,n^8$.

32. $x^6-26\,x^3-27$.

33. $\cancel{(x+2\,y)^3+(3\,x-y)^3}$.

34. $(a+2\,x)^2+10\,(a+2\,x)-144$.

35. $27\,x^2-75\,y^2-120\,yz-48\,z^2$.

36. $(a^2+4\,ab+b^2)^2-(a^2+b^2)^2$.

37. $(16\,m^2+n^2)^2-64\,m^2n^2$.

38. $49\,a^2+4-36\,b^2-28\,a$.

39. $49\,a^6b^2+12\,a^4b^6+4\,a^2b^{10}$.

40. $14\,x^2-25\,x+6$.

41. $a^{14}-x^{14}$.

42. $x^{14}-2\,x^7+1$.

43. $9\,a^2c^2-16\,a^2d^2-36\,b^2c^2+64\,b^2d^2$.

44. $m^9n^3-243\,m^4n^8$.

45. $-7\,x^2-26\,x+8$.

46. $a^7+128\,b^7$.

47. $48\,x^3y-52\,x^2y^2-140\,xy^3$.

48. Resolve a^4-81 into two factors, one of which is $a-3$.

49. Resolve x^6-64 into two factors, one of which is $x+2$.

50. Resolve x^9-y^9 into two factors, one of which is $x-y$.

51. Resolve a^9+1 into two factors, one of which is $a+1$.

52. Resolve $1+x^9$ into three factors by the method of Case VII.

53. Resolve a^9-512 into three factors.

Factor the following:

54. a^2-m^2+a+m.

55. $(x^2+4\,x)^2-37\,(x^2+4\,x)+160$.

56. $n^{10}-1024$.

57. m^3+m+x^3+x.

58. $a^2c^3 - 4\,b^2c^3 - 8\,a^2d^3 + 32\,b^2d^3$.

59. $(m-n)(x^2-y^2)+(x+y)(m^2-n^2)$.

60. $(x-1)^3+6\,(x-1)^2+9\,(x-1)$.

61. $(m+n)(m^2-x^2)-(m+x)(m^2-n^2)$.

62. $a^2-4\,b^2-a-2\,b$.

63. $(x^2+4\,y^2-z^2)^2-16\,x^2y^2$.

64. $(x^2-9\,x)^2+4\,(x^2-9\,x)-140$.

65. $a^3b^3+27\,a^3y^3-8\,b^3x^3-216\,x^3y^3$.

66. $(m^2+m)^2+2\,(m^2+m)(m+1)+(m+1)^2$.

67. $(2\,x^2-3)^2-x^2$.

68. $64\,a^3x^3+8\,a^3-8\,x^3-1$.

69. $(4\,a^2-b^2-9)^2-36\,b^2$.

70. $(x+2\,y)^3-x\,(x^2-4\,y^2)$.

71. $16\,x^2+y^2-25\,z^2-1+8\,xy+10\,z$.

72. $(a^2+6\,a+8)^2-14\,(a^2+6\,a+8)-15$.

73. $(1+x^3)+(1+x)^3$.

74. $a^4-9+2\,a\,(a^2+3)$.

75. $(x^3+y^3)-xy\,(x+y)$.

76. $(a^3-8\,m^3)-a\,(a-2\,m)^2$.

77. $9\,a^2(3\,a+2)^2+6\,a\,(3\,a+2)+1$.

78. $m^8-m^5+32\,m^3-32$.

79. $a\,(a-c)-b\,(b-c)$.

80. $m^2(m+p)+n^2(n-p)$.

81. $x^9+8\,x^6+x^3+8$.

82. $(27\,m^3-x^3)+(3\,m+x)(9\,m^2-6\,mx+x^2)$.

83. $(4\,a^2+9)^2-24\,a\,(4\,a^2+9)+144\,a^2$.

84. $16\,a^2+9\,b^2-25\,c^2-4\,d^2-24\,ab-20\,cd$.

85. $m^9+m^6-64\,m^3-64$.

86. $(x^2+y^2)^3-4\,x^2y^2\,(x^2+y^2)$.

87. $a^5+a^4b+a^3b^2+a^2b^3+ab^4+b^5$.

88. $(8\,n^3-27)+(2\,n-3)(4\,n^2+4\,n-6)$.

89. $x^3+2\,x^2+2\,x+1$.

(By altering the order of the terms, this may be written

$$x^3+1+(2\,x^2+2\,x),\ \text{or}\ (x+1)(x^2-x+1)+2\,x\,(x+1),$$

and $x+1$ is a factor of the given expression.)

90. $x^3 - 3x^2 + 3x - 1.$

91. $a^3 + 3a^2b + 6ab^2 + 8b^3.$

92. $8x^3 + 36x^2y + 54xy^2 + 27y^3.$

Additional methods in factoring will be found in §§ 298 to 300, and in Chapter XXXIV.

124. By § 54, $(+a) \times (+b) = +ab$, $(+a) \times (-b) = -ab$,

$$(-a) \times (+b) = -ab, \quad (-a) \times (-b) = +ab.$$

Hence, in the indicated product of two factors, *the signs of both factors may be changed without altering the product; but if the sign of either one be changed, the sign of the product will be changed.*

If either factor is a polynomial, care must be taken, on changing its sign, to change the sign of *each of its terms.*

Thus, the result of Ex. 3, § 107, may be written in the forms

$$(b-a)(n-m), \quad -(b-a)(m-n), \text{ or } -(a-b)(n-m).$$

In like manner, in the indicated product of more than two factors, *the signs of any* **even** *number of them may be changed without altering the product; but if the signs of any* **odd** *number of them be changed, the sign of the product will be changed* (§ 55).

Thus, $(a-b)(c-d)(e-f)$ may be written in the forms

$$(a-b)(d-c)(f-e),$$
$$(b-a)(c-d)(f-e),$$
$$-(b-a)(d-c)(f-e), \text{ etc.}$$

SOLUTION OF EQUATIONS BY FACTORING

125. Let it be required to solve the equation

$$(x-3)(2x+5) = 0.$$

It is evident that the equation will be satisfied when x has such a value that *one* of the factors of the first member is equal to zero; for if any factor of a product is equal to zero, the product is equal to zero.

Hence, the equation will be satisfied when x has such a value that either

$$x - 3 = 0, \qquad (1)$$

or

$$2x + 5 = 0. \qquad (2)$$

Solving (1) and (2), we have $x = 3$ or $-\frac{5}{2}$.

It will be observed that the roots are obtained by *placing the factors of the first member separately equal to zero, and solving the resulting equations.*

126. Examples.

1. Solve the equation $x^2 - 5x - 24 = 0$.

Factoring the first member, $(x - 8)(x + 3) = 0$. (§ 116)

Placing the factors separately equal to 0 (§ 125), we have

$$x - 8 = 0, \text{ whence } x = 8;$$

and

$$x + 3 = 0, \text{ whence } x = -3.$$

2. Solve the equation $4x^2 - 2x = 0$.

Factoring the first member, $2x(2x - 1) = 0$.

Placing the factors separately equal to 0, we have

$$2x = 0, \text{ whence } x = 0;$$

and

$$2x - 1 = 0, \text{ whence } x = \frac{1}{2}.$$

3. Solve the equation $x^3 + 4x^2 - x - 4 = 0$.

Factoring the first member, we have by § 108,

$$(x + 4)(x^2 - 1) = 0, \text{ or } (x + 4)(x + 1)(x - 1) = 0.$$

Then,

$$x + 4 = 0, \text{ whence } x = -4;$$

$$x + 1 = 0, \text{ whence } x = -1;$$

and

$$x - 1 = 0, \text{ whence } x = 1.$$

4. Solve the equation $x^3 - 27 - (x^2 + 9x - 36) = 0$.

Factoring the first member, we have by §§ 116 and 121,

$$(x - 3)(x^2 + 3x + 9) - (x - 3)(x + 12) = 0.$$

Or,

$$(x - 3)(x^2 + 3x + 9 - x - 12) = 0.$$

Or,

$$(x - 3)(x^2 + 2x - 3) = 0.$$

Or, $$(x-3)(x+3)(x-1)=0.$$

Placing the factors separately equal to 0, $x = 3$, -3, or 1.

The pupil should endeavor to put down the values of x without actually placing the factors equal to 0, as shown in Ex. 4.

EXERCISE 43

Solve the following equations:

1. $x^2+7x=0.$
2. $5x^3-4x^2=0.$
3. $3x^3-108x=0.$
4. $(3x-2)(4x^2-25)=0.$
5. $x^2-15x+54=0.$
6. $x^2+23x+102=0.$
7. $x^2+4x-96=0.$
8. $x^2-x-110=0.$
9. $x^2+ax-2a^2=0.$
10. $(5x+1)(x^2-6x-91)=0.$
11. $x^4+18x^3+32x^2=0.$
12. $x^4-13x^2+36=0.$
13. $8x^2-10x+3=0.$
14. $6x^2+7x+2=0.$
15. $3x^2-mx-4m^2=0.$
16. $10x^2+7x-12=0.$
17. $15x^2+x-2=0.$
18. $12x^3-29x^2+15x=0.$
19. $x^2-ax+bx-ab=0.$
20. $x^2+mx+nx+mn=0.$
21. $x^2-2cx-8x+16c=0.$
22. $x^2+3m^2x-5m^3x-15m^5=0.$
23. $27x^3+18x^2-3x-2=0.$
24. $(x-2)^2-4(x-2)+3=0.$
25. $(4x^2-49)(x^2-3x-10)(8x^2+14x-15)=0.$
26. $(x-2)(5x^2+8x-4)-(x^2-4)=0.$
27. $(x^2-1)(x^2-9)+3(x-1)(x+3)=0.$

IX. HIGHEST COMMON FACTOR. LOWEST COMMON MULTIPLE

(We consider in the present chapter the Highest Common Factor and Lowest Common Multiple of *Monomials*, or of polynomials which can be readily factored by inspection.

The Highest Common Factor and Lowest Common Multiple of polynomials which cannot be readily factored by inspection, are considered in §§ 439 to 443.)

HIGHEST COMMON FACTOR

127. The **Highest Common Factor** (H. C. F.) of two or more expressions is their common factor of *highest degree* (§ 64).

If several common factors are of equally high degree, it is understood that *the* highest common factor is the one having the numerical coefficient of greatest absolute value in its term of highest degree.

For example, if the common factors were $6x - 3$ and $2x - 1$, the former would be the H. C. F.

128. Two expressions are said to be *prime to each other* when unity is their highest common factor.

129. CASE I. **Highest Common Factor of Monomials.**

Ex. Required the H. C. F. of $42\,a^3b^2$, $70\,a^2bc$, and $98\,a^4b^3d^2$.

By the rule of Arithmetic, the H. C. F. of 42, 70, and 98 is 14.

It is evident by inspection that the expression of highest degree which will exactly divide a^3b^2, a^2bc, and $a^4b^3d^2$ is a^2b.

Then, the H. C. F. of the given expressions is $14\,a^2b$.

It will be observed, in the above result, that *the exponent of each letter is the lowest exponent with which it occurs in any of the given expressions.*

EXERCISE 44

Find the H. C. F. of the following:

1. $14\,x^2y$, $21\,xy^4$.
2. $64\,a^5b^3$, $112\,b^4c^4$.

3. $60(x-y)^5$, $84(x-y)^4$.

4. $108\,m^3n^3p^3$, $90\,m^2np^7$.

5. $72\,a^5b^3$, $27a^3b^2$, $99\,a^4b^4$.

6. $44\,x^7yz^6$, $88\,x^6y^3z^4$, $110\,x^5y^5z^7$.

7. $32\,a^4x^4$, $128\,a^6b^2x^3$, $192\,a^3x^2y^3$.

8. $136\,a^4m^5n^3$, $51\,b^3mn^6$, $119\,c^2m^3n^9$.

9. $72\,x^4y^5z^5$, $168\,x^5y^2z^7$, $120\,x^6y^3z^3$.

10. $26(a+b)^2(a-b)^6$, $91(a+b)^5(a-b)^3$.

130. CASE II. **Highest Common Factor of Polynomials which can be readily factored by Inspection.**

1. Required the H. C. F. of

$$5\,x^4y-45\,x^2y \text{ and } 10\,x^3y^2-40\,x^2y^2-210\,xy^2.$$

By §§ 107, 114, and 116, $5\,x^4y-45\,x^2y = 5\,x^2y(x^2-9)$

$$= 5\,x^2y(x+3)(x-3); \qquad (1)$$

and $\quad 10\,x^3y^2-40\,x^2y^2-210\,xy^2 = 10\,xy^2(x^2-4\,x-21)$

$$= 10\,xy^2(x-7)(x+3). \qquad (2)$$

The H. C. F. of the numerical coefficients 5 and 10 is 5.

It is evident by inspection that the H. C. F. of the literal portions of the expressions (1) and (2) is $xy(x+3)$.

Then, the H. C. F. of the given expressions is $5\,xy(x+3)$.

It is sometimes necessary to change the form of the factors in finding the H. C. F. of expressions.

2. Find the H. C. F. of $a^2+2\,a-3$ and $1-a^3$.

By § 116, $\quad a^2+2\,a-3=(a-1)(a+3)$.

By § 121, $\quad 1-a^3=(1-a)(1+a+a^2)$.

By § 124, the factors of the first expression can be put in the form

$$-(1-a)(3+a).$$

Hence, the H. C. F. is $1-a$.

EXERCISE 45

Find the H. C. F. of the following:

1. $30\,x^4y^3+10\,x^3y^4$, $15\,x^2y^4-30\,xy^5$.

2. $a^2-16b^2,\ a^2+8ab+16b^2.$

3. $m^2-14m+45,\ m^2-10m+25.$

4. $x^3-5x^2+3x-15,\ 4x^7+12x^5.$

5. $a^3+64,\ a^2-7a-44.$

6. $9-x^2,\ x^2-x-6.$

7. $ac-bc-ad+bd,\ d^2-c^2.$

8. $x^2+13x+22,\ 2x^2+9x+10.$

9. $3ac-4ad-6bc+8bd,\ a^2+7ab-18b^2.$

10. $x^2+y^2-z^2-2xy,\ x^2-y^2-z^2+2yz.$

11. $3x^2-16xy+5y^2,\ x^2+10xy-75y^2.$

12. $m^5-8m^2,\ m^4+4m^2+16.$

13. $2x^2-7x+6,\ 6x^2-11x+3.$

14. $2x^2-13xy+6y^2,\ xy^2-4x^3.$

15. $1-11a+18a^2,\ 8a^3-1,\ 18a^2-5a-2.$

16. $8a^3-26a^2b+20ab^2,\ 12a^3-10a^2b-28ab^2.$

17. $x^2+18x+77,\ x^2+22x+121,\ x^2+x-110.$

18. $16m^2-9n^2,\ 16m^2-24mn+9n^2,\ 9mn^2-12m^2n.$

19. $x^3-27,\ x^2-6x+9,\ 2ax-6a-bx+3b.$

20. $27a^3+8b^3,\ 9a^2-4b^2,\ 9a^2+12ab+4b^2.$

21. $a^2-3a-18,\ 2a^2-a-21,\ 3a^2+4a-15.$

22. $2x^3-12x^2+16x,\ 3x^4-3x^3-36x^2,\ 5x^5+5x^4-100x^3.$

23. $125m^4-8m,\ 10m^3+m^2-2m,\ 25m^3-20m^2+4m.$

24. $a^4+3a^2-40,\ a^4-25,\ a^3+a^2-5a-5.$

25. $2x^3-x^2-6x+3,\ 6x^2-19x+8,\ 4x^2+8x-5.$

26. $a^2-(b+c)^2,\ (b-a)^2-c^2,\ b^2-(a-c)^2.$

27. $8x^5y+x^2y^4,\ 64x^6y^2+2xy^7,\ 24x^3y-30x^2y^2-21xy^3.$

28. $2a^2+17a+36,\ 4a^2-4a-99,\ 6a^2+25a-9.$

LOWEST COMMON MULTIPLE

131. A **Common Multiple** of two or more expressions is an expression which is exactly divisible by each of them.

132. The **Lowest Common Multiple** (L. C. M.) of two or more expressions is their common multiple of *lowest degree.*

If several common multiples are of equally low degree, it is understood that *the* lowest common multiple is the one having the numerical coefficient of least absolute value in its term of highest degree.

For example, if the common multiples were $4x-2$ and $6x-3$, the former would be the L. C. M.

133. Case I. **Lowest Common Multiple of Monomials.**

Ex. Required the L. C. M. of $36a^3x$, $60a^2y^2$, and $84cx^3$.

By the rule of Arithmetic, the L. C. M. of 36, 60, and 84 is 1260.

It is evident by inspection that the expression of lowest degree which is exactly divisible by a^3x, a^2y^2, and cx^3 is $a^3cx^3y^2$.

Then, the L. C. M. of the given expressions is $1260a^3cx^3y^2$.

It will be observed, in the above result, that *the exponent of each letter is the highest exponent with which it occurs in any of the given expressions.*

EXERCISE 46

Find the L. C. M. of the following:

1. $5x^3y^3$, $6x^2y^4$.
2. $18a^6b$, $45b^5c$.
3. $28x^5$, $36y^4$.
4. $42m^4n^2$, $98n^5p^3$.
5. $105a^2b$, $70b^2c$, $63c^2a$.
6. $50x^4y^5$, $24x^5y^3$, $40x^6y^4$.
7. $21ab^4$, $35b^2c^6$, $91a^3c^3$.
8. $56a^2b^3$, $84bx^5$, $48x^4y^6$.
9. $60a^3bc^2$, $75a^5b^6d$, $90a^4c^7d^8$.
10. $99m^4nx^8$, $66m^3n^9y^5$, $165n^5x^4y^7$.

134. Case II. **Lowest Common Multiple of Polynomials which can be readily factored by Inspection.**

1. Required the L. C. M. of

$$x^2-5x+6,\ x^2-4x+4, \text{ and } x^3-9x.$$

By § 116, $x^2 - 5x + 6 = (x - 3)(x - 2)$.

By § 113, $x^2 - 4x + 4 = (x - 2)^2$.

By § 114, $x^3 - 9x = x(x + 3)(x - 3)$.

It is evident by inspection that the L. C. M. of these expressions is

$$x(x - 2)^2(x + 3)(x - 3).$$

It is sometimes necessary to change the form of the factors.

2. Find the L. C. M. of $ac - bc - ad + bd$ and $b^2 - a^2$.

By § 108, $ac - bc - ad + bd = (a - b)(c - d)$.

By § 114, $b^2 - a^2 = (b + a)(b - a)$.

By § 124, the factors of the first expression can be written

$$(b - a)(d - c).$$

Hence, the L. C. M. is $(b + a)(b - a)(d - c)$, or $(b^2 - a^2)(d - c)$.

EXERCISE 47

Find the L. C. M. of the following:

1. $x^2 - 4y^2,\ x^2 + 4xy + 4y^2$.
2. $a^3b + 2a^2b^2,\ 2a^2b^2 + ab^3$.
3. $m^2 - 6m + 9,\ m^2 - 11m + 24$.
4. $a^4 - 49a^2b^2,\ a^5 + 12a^4b + 35a^3b^2$.
5. $2x^3 + 2x^2 - 84x,\ 3x^3 - 3x^2 - 90x$.
6. $a^3 - x^3,\ a^3 - a^2x + ax^2 - x^3$.
7. $1 + 27x^3,\ 1 - 5x - 24x^2$.
8. $ac - 3ad - 2bc + 6bd,\ 3ac + ad - 6bc - 2bd$.
9. $x^2 - y^2 - z^2 + 2yz,\ x^2 - y^2 + z^2 + 2xz$.
10. $a^2 - 7a + 10,\ 10 - 5a + 2a^2 - a^3$.
11. $x^3 + 8,\ 4x^2 - (x^2 + 4)^2$.
12. $2x^2 + 3x - 35,\ 2x^2 + 19x + 45$.
13. $9n^2 - 27n + 8,\ 3n^2 - 2n - 16$.

14. $16x^2 - 25y^2,\ 12x^2 + 15xy,\ 8xy - 10y^2.$

15. $x^2 - 15x + 50,\ x^2 + 2x - 35,\ x^2 - 3x - 70.$

16. $a^2 - 4ab + 4b^2,\ a^3 - 8b^3,\ a^3b + 2a^2b^2 + 4ab^3.$

17. $m^2 - 10mn + 21n^2,\ m^2 - 5mn - 24n^2,\ m^4 - 81n^4.$

18. $x^2 + 5x + 6,\ x^2 - 2x - 8,\ x^3 + 2x^2 + 5x + 10.$

19. $9ab^3 - 4a^3b,\ 8ac + 2ad - 12bc - 3bd.$

20. $a^5 - 16a,\ a^5 - 3a^3 - 4a,\ a^5 + 5a^3 + 4a.$

21. $27n^4 + 64n,\ 18n^4 - 32n^2,\ 9n^5 + 21n^4 + 12n^3.$

22. $9x^2 + 30x + 25,\ 6x^2 + 7x - 5,\ 10x^2 - 9x + 2.$

23. $n^2 - 5n + 6,\ 9n^2 - n^4,\ 10 - n - 2n^2.$

24. $x^3 - y^3,\ x^2 - 2xy + y^2,\ x^4 + x^2y^2 + y^4.$

25. $3ac + ad - 6bc - 2bd,\ ac - 4ad - 2bc + 8bd,\ 3c^2 - 11cd - 4d^2.$

26. $2x^2 - x - 15,\ 2x^2 - 7x + 3,\ 2x^2 - 9x + 9.$

27. $a^2 + 4b^2 - 9c^2 - 4ab,\ a^2 - 4b^2 - 9c^2 + 12bc,\ a^2 - 4b^2 + 9c^2 - 6ac.$

28. $3m^3 + m^2n - 2mn^2,\ 6m^2n + 11mn^2 + 5n^3,$
$9m^3n + 5m^2n^2 - 4mn^3.$

29. $32a^6 + 4a^3,\ 12a^4 + 12a^3 + 3a^2,\ 32a^5 + 8a^3 + 2a.$

X. FRACTIONS

135. The quotient of a divided by b is written $\frac{a}{b}$ (§ 6).

The expression $\frac{a}{b}$ is called a **Fraction**; the dividend a is called the *numerator*, and the divisor b the *denominator*.

The numerator and denominator are called the *terms* of the fraction.

136. It follows from § 69, (3), that

If the terms of a fraction be both multiplied, or both divided, by the same expression, the value of the fraction is not changed.

137. By the Rule of Signs in Division (§ 68),

$$\frac{+a}{+b}=\frac{-a}{-b}=-\frac{+a}{-b}=-\frac{-a}{+b}.$$

That is, *if the signs of both terms of a fraction be changed, the sign before the fraction is not changed; but if the sign of either one be changed, the sign before the fraction is changed.*

If either term is a polynomial, care must be taken, on changing its sign, to change the sign of *each of its terms.*

Thus, the fraction $\frac{a-b}{c-d}$, by changing the signs of both numerator and denominator, can be written $\frac{b-a}{d-c}$ (§ 51).

138. It follows from §§ 124 and 137 that *if either term of a fraction is the indicated product of two or more expressions, the signs of any* **even** *number of them may be changed without changing the sign before the fraction; but if the signs of any* **odd** *number of them be changed, the sign before the fraction is changed.*

Thus, the fraction $\frac{a-b}{(c-d)(e-f)}$ may be written

$$\frac{a-b}{(d-c)(f-e)},\ \frac{b-a}{(d-c)(e-f)},\ -\frac{b-a}{(d-c)(f-e)},\ \text{etc.}$$

EXERCISE 48

Write each of the following in three other ways without changing its value:

1. $\frac{a}{2}$. 2. $\frac{n+3}{7}$. 3. $\frac{8}{2-x}$. 4. $\frac{2x-7}{x+2}$. 5. $\frac{6x-5}{(x-3)(y+4)}$.

6. Write $\frac{(3x-1)(a-4)}{(x+5)(y-2)}$ in four other ways without changing its value.

REDUCTION OF FRACTIONS

139. Reduction of a Fraction to its Lowest Terms.

A fraction is said to be in its *lowest terms* when its numerator and denominator are prime to each other (§ 128).

(We consider in the present chapter those cases only in which the numerator and denominator can be readily factored by inspection.

The cases in which the numerator and denominator cannot be readily factored by inspection are considered in § 444.)

140. By § 136, dividing both terms of a fraction by the same expression, or cancelling common factors in the numerator and denominator, does not alter the value of the fraction.

We then have the following rule:

Resolve both numerator and denominator into their factors, and cancel all that are common to both.

1. Reduce $\frac{24a^4b^2cx}{40a^2b^2c^2d^3}$ to its lowest terms.

We have, $\frac{24a^4b^2cx}{40a^2b^2c^2d^3}=\frac{2^3\times 3\times a^4b^2cx}{2^3\times 5\times a^2b^2c^2d^3}=\frac{3a^2x}{5cd^3}$,

by cancelling the common factor $2^3\times a^2b^2c$.

2. Reduce $\frac{x^3-27}{x^2-2x-3}$ to its lowest terms.

By §§ 121 and 116, $\frac{x^3-27}{x^2-2x-3}=\frac{(x-3)(x^2+3x+9)}{(x-3)(x+1)}=\frac{x^2+3x+9}{x+1}$.

3. Reduce $\frac{ax-bx-ay+by}{b^2-a^2}$ to its lowest terms.

By §§ 108 and 114, $\dfrac{ax - bx - ay + by}{b^2 - a^2} = \dfrac{(a-b)(x-y)}{(b+a)(b-a)}.$

By § 138, the signs of the terms of the factors of the numerator can be changed without altering the value of the fraction; and in this way the first factor of the numerator becomes the same as the second factor of the denominator.

Then, $\dfrac{ax - bx - ay + by}{b^2 - a^2} = \dfrac{(b-a)(y-x)}{(b+a)(b-a)} = \dfrac{y-x}{b+a}.$

If all the factors of the numerator are cancelled, 1 remains to form a numerator; if all the factors of the denominator are cancelled, it is a case of exact division.

EXERCISE 49

Reduce each of the following to its lowest terms:

1. $\dfrac{5x^4y^5z^2}{3xy^5z^2}.$
2. $\dfrac{12a^5b^4}{42b^2c^3}.$
3. $\dfrac{54mn^3}{99m^5n^3}.$
4. $\dfrac{63x^3y^4z^7}{84x^5y^4z^2}.$
5. $\dfrac{126a^6b^8c^5}{14a^6c^4}.$
6. $\dfrac{26m^3n^4p^2}{130m^4n^6p^7}.$
7. $\dfrac{90a^3m^7n^4}{36am^7n^8}.$
8. $\dfrac{88x^4y^6z^3}{66x^5yz^9}.$
9. $\dfrac{120a^7b^4c^{10}}{75ab^9c^2}.$
10. $\dfrac{15x^4y + 10x^3y^2}{6x^3y^4 + 4x^2y^5}.$
11. $\dfrac{x^2 - 9x + 18}{x^2 + x - 12}.$
12. $\dfrac{a^2 + 11ab + 28b^2}{a^3 + 14a^2b + 49ab^2}.$
13. $\dfrac{64x^3 + 136x^2y + 72xy^2}{64x^2y - 81y^3}.$
14. $\dfrac{m^4 + m^3n - 56m^2n^2}{m^3 - 64mn^2}.$
15. $\dfrac{a^3 + b^3}{a^2 - 2ab - 3b^2}.$
16. $\dfrac{ac + 3ad + 2bc + 6bd}{3ac - ad + 6bc - 2bd}.$
17. $\dfrac{8x^3 - 125}{2x^3 + x^2 - 15x}.$
18. $\dfrac{a^2 + a - 12}{3a^2 - 13a + 12}.$
19. $\dfrac{(x^2 - 49)(x^2 - 16x + 63)}{(x^2 - 14x + 49)(x^2 - 2x - 63)}.$
20. $\dfrac{3a^3 - 4a^2 - 3a + 4}{9a^3 + 9a^2 - 16a - 16}.$
21. $\dfrac{4m^2 + 16mn + 15n^2}{6m^2 - mn - 15n^2}.$
22. $\dfrac{16x^4 + 4x^2 + 1}{8x^3 - 1}.$
23. $\dfrac{x^2 - 9y^2 - z^2 + 6yz}{x^2 - 9y^2 + z^2 - 2xz}.$
24. $\dfrac{(a - 2b)^2 - (3c - d)^2}{(a + d)^2 - (2b + 3c)^2}.$
25. $\dfrac{a^6 + 28a^3b^3 + 27b^6}{a^4 + 9a^2b^2 + 81b^4}.$
26. $\dfrac{25 - x^2}{x^2 - 11x + 30}.$
27. $\dfrac{9x^2 - 49y^2}{28xy^2 - 12x^2y}.$

28. $\dfrac{8b^3-a^3}{a^2+5ab-14b^2}$.

30. $\dfrac{x^2-(y-z)^2}{z^2-(x+y)^2}$.

29. $\dfrac{27-x^3}{4x^2-9x-9}$.

31. $\dfrac{21-x-10x^2}{15xy-20x-21y+28}$.

141. Reduction of a Fraction to an Integral or Mixed Expression.

A **Mixed Expression** is a polynomial consisting of a rational and integral expression (§ 63), with one or more fractions.

Thus, $a+\dfrac{b}{c}$, and $\dfrac{x}{3}+\dfrac{2x-y}{x-y}$ are mixed expressions.

142. A fraction may be reduced to an integral or mixed expression by the operation of division, if the degree (§ 64) of the numerator is equal to, or greater than, that of the denominator.

1. Reduce $\dfrac{6x^2+15x-2}{3x}$ to a mixed expression.

By § 72, $\dfrac{6x^2+15x-2}{3x}=\dfrac{6x^2}{3x}+\dfrac{15x}{3x}-\dfrac{2}{3x}=2x+5-\dfrac{2}{3x}$.

2. Reduce $\dfrac{12x^3-8x^2+4x-5}{4x^2+3}$ to a mixed expression.

$$\begin{array}{r}
4x^2+3)\,12x^3-8x^2+4x-5\,(3x-2\\
\underline{12x^3\qquad\quad+9x}\\
-8x^2-5x\\
\underline{-8x^2\qquad\quad-6}\\
-5x+1
\end{array}$$

Since the dividend is equal to the product of the divisor and quotient, plus the remainder, we have

$$12x^3-8x^2+4x-5=(4x^2+3)(3x-2)+(-5x+1).$$

Dividing both members by $4x^2+3$, we have

$$\frac{12x^3-8x^2+4x-5}{4x^2+3}=3x-2+\frac{-5x+1}{4x^2+3}.$$

Thus, a remainder of lower degree than the divisor may be written over the divisor in the form of a fraction, and the result added to the quotient.

If the first term of the numerator is negative, as in Ex. 2, it is usual to *change the sign of each term of the numerator*, changing the sign before the fraction (§ 137).

Thus, $$\frac{12x^3-8x^2+4x-5}{4x^2+3}=3x-2-\frac{5x-1}{4x^2+3}.$$

EXERCISE 50

Reduce each of the following to a mixed expression:

1. $\frac{15m^2+12m-4}{3m}$.
2. $\frac{30a^6-5a^4+15a^2+7}{5a^2}$.
3. $\frac{9x^2+2}{3x-1}$.
4. $\frac{x^3-y^3}{x+y}$.
5. $\frac{a^3+8b^3}{a-2b}$.
6. $\frac{49x^2}{7x+3y}$.
7. $\frac{a^4+b^4}{a-b}$.
8. $\frac{m^5-n^5}{m+n}$.
9. $\frac{14a^3+39a^2+4a-19}{2a+5}$.
10. $\frac{35x^2+10x+3}{5x+2}$.
11. $\frac{3a^3+8a^2-4}{a^2+2a-3}$.
12. $\frac{15x^4-6x^3-20x^2-7}{3x^2-4}$.
13. $\frac{24x^3+21x+19}{4x^2-2x+5}$.
14. $\frac{6a^4-17a^3b-21a^2b^2+19ab^3+22b^4}{2a^2-5ab-6b^2}$.

143. Reduction of Fractions to their Lowest Common Denominator.

To reduce fractions to their **Lowest Common Denominator** (L. C. D.) is to express them as equivalent fractions, each having for a denominator the L. C. M. of the given denominators.

Let it be required to reduce $\frac{4cd}{3a^2b^3}$, $\frac{3m}{2ab^2}$, and $\frac{5n}{4a^3b}$ to their lowest common denominator.

The L. C. M. of $3a^2b^3$, $2ab^2$, and $4a^3b$ is $12a^3b^3$ (§ 133).

By § 136, if the terms of a fraction be both multiplied by the same expression, the value of the fraction is not changed.

Multiplying both terms of $\frac{4cd}{3a^2b^3}$ by $4a$, both terms of $\frac{3m}{2ab^2}$ by $6a^2b$, and both terms of $\frac{5n}{4a^3b}$ by $3b^2$, we have

$$\frac{16acd}{12a^3b^3},\ \frac{18a^2bm}{12a^3b^3},\ \text{and}\ \frac{15b^2n}{12a^3b^3}.$$

It will be seen that the terms of each fraction are multiplied by an expression, which is obtained by dividing the L. C. D. by the denominator of this fraction.

Whence the following rule.

Find the L. C. M. *of the given denominators.*

Multiply both terms of each fraction by the quotient obtained by dividing the L. C. D. *by the denominator of this fraction.*

Before applying the rule, each fraction should be reduced to its lowest terms.

144. *Ex.* Reduce $\frac{4a}{a^2-4}$ and $\frac{3a}{a^2-5a+6}$ to their lowest common denominator.

We have, $$a^2-4=(a+2)(a-2),$$

and $$a^2-5a+6=(a-2)(a-3).$$

Then, the L. C. D. is $(a+2)(a-2)(a-3)$. (§ 134)

Dividing the L. C. D. by $(a+2)(a-2)$, the quotient is $a-3$; dividing it by $(a-2)(a-3)$, the quotient is $a+2$.

Then, by the rule, the required fractions are

$$\frac{4a(a-3)}{(a+2)(a-2)(a-3)} \text{ and } \frac{3a(a+2)}{(a+2)(a-2)(a-3)}.$$

EXERCISE 51

Reduce the following to their lowest common denominator:

1. $\frac{7ab}{6}, \frac{3bc}{10}, \frac{2ca}{15}$.

2. $\frac{5}{2m^2n}, \frac{4}{5m^3n^3}, \frac{6}{7mn^2}$.

3. $\frac{3x+4z}{22xy^3}, \frac{6x-5y}{33yz^2}$.

4. $\frac{11c^4p}{12a^2b}, \frac{9a^5m}{14b^4c}, \frac{8b^3n}{21c^3a}$.

5. $\frac{4a^2}{4a^2-9}, \frac{2}{6a^2-9a}$.

6. $\frac{1}{m-n}, \frac{3mn}{2(m-n)^2}, \frac{2m^2n^2}{3(m-n)^3}$.

7. $\frac{3n}{n^3-8}, \frac{5}{n^2-4n+4}$.

8. $\frac{2}{a^3+3a^2+2a+6}, \frac{3a}{a^3+27}$.

9. $\frac{2}{x+2}, \frac{4}{x-2}, \frac{6}{x^2-3}$.

10. $\dfrac{a+3b}{a^2-7ab+12b^2}, \dfrac{a-3b}{a^2-ab-12b^2}, \dfrac{a+4b}{a^2-9b^2}.$

11. $\dfrac{2x+3}{x^2+3x-10}, \dfrac{x+2}{2x^2+7x-15}, \dfrac{x-5}{2x^2-7x+6}.$

ADDITION AND SUBTRACTION OF FRACTIONS

145. By § 72, $\dfrac{b}{a}+\dfrac{c}{a}-\dfrac{d}{a}=\dfrac{b+c-d}{a}.$

We then have the following rule:

To add or subtract fractions, reduce them, if necessary, to equivalent fractions having the lowest common denominator.

Add or subtract the numerator of each resulting fraction, according as the sign before the fraction is + or −, and write the result over the lowest common denominator.

The final result should be reduced to its lowest terms.

146. Examples.

1. Simplify $\dfrac{4a+3}{4a^2b}+\dfrac{1-6b^2}{6ab^3}.$

The L. C. D. is $12a^2b^3$; multiplying the terms of the first fraction by $3b^2$, and the terms of the second by $2a$, we have

$$\frac{4a+3}{4a^2b}+\frac{1-6b^2}{6ab^3}=\frac{12ab^2+9b^2}{12a^2b^3}+\frac{2a-12ab^2}{12a^2b^3}$$

$$=\frac{12ab^2+9b^2+2a-12ab^2}{12a^2b^3}=\frac{9b^2+2a}{12a^2b^3}.$$

If a fraction whose numerator is a polynomial is preceded by a − sign, it is convenient to enclose the numerator in parentheses preceded by a − sign, as shown in the last term of the numerator in equation (A), of Ex. 2.

If this is not done, care must be taken to *change the sign of each term of the numerator* before combining it with the other numerators.

2. Simplify $\dfrac{5x-4y}{6}-\dfrac{7x-2y}{14}.$

The L. C. D. is 42; whence,

$$\frac{5x-4y}{6}-\frac{7x-2y}{14}=\frac{35x-28y}{42}-\frac{21x-6y}{42}$$

$$=\frac{35x-28y-(21x-6y)}{42} \qquad \text{(A)}$$

$$=\frac{35x-28y-21x+6y}{42}=\frac{14x-22y}{42}=\frac{7x-11y}{21}.$$

3. Simplify $\dfrac{1}{x^2+x}-\dfrac{1}{x^2-x}$.

We have, $x^2+x=x(x+1)$, and $x^2-x=x(x-1)$.

Then, the L. C. D. is $x(x+1)(x-1)$, or $x(x^2-1)$.

Multiplying the terms of the first fraction by $x-1$, and the terms of the second by $x+1$, we have

$$\frac{1}{x^2+x}-\frac{1}{x^2-x}=\frac{x-1}{x(x^2-1)}-\frac{x+1}{x(x^2-1)}$$

$$=\frac{x-1-(x+1)}{x(x^2-1)}=\frac{x-1-x-1}{x(x^2-1)}=\frac{-2}{x(x^2-1)}.$$

By changing the sign of the numerator, at the same time changing the sign before the fraction (§ 137), we may write the answer $-\dfrac{2}{x(x^2-1)}$.

Or, by changing the sign of the numerator, and of the factor x^2-1 of the denominator (§ 138), we may write it $\dfrac{2}{x(1-x^2)}$.

4. Simplify $\dfrac{1}{a^2-3a+2}-\dfrac{2}{a^2-4a+3}+\dfrac{1}{a^2-5a+6}$.

We have, $a^2-3a+2=(a-1)(a-2)$, $a^2-4a+3=(a-1)(a-3)$, and $a^2-5a+6=(a-2)(a-3)$.

Then, the L. C. D. is $(a-1)(a-2)(a-3)$.

Whence, $\dfrac{1}{a^2-3a+2}-\dfrac{2}{a^2-4a+3}+\dfrac{1}{a^2-5a+6}$

$$=\frac{a-3}{(a-1)(a-2)(a-3)}-\frac{2(a-2)}{(a-1)(a-2)(a-3)}+\frac{a-1}{(a-1)(a-2)(a-3)}$$

$$=\frac{a-3-2(a-2)+a-1}{(a-1)(a-2)(a-3)}=\frac{a-3-2a+4+a-1}{(a-1)(a-2)(a-3)}$$

$$=\frac{0}{(a-1)(a-2)(a-3)}=0.$$

EXERCISE 52

Simplify the following:

1. $\frac{4x+7}{10}+\frac{6x-5}{15}$.

2. $\frac{3}{2a^2b^3}+\frac{5}{7a^4b}$.

3. $\frac{4a-9}{9}-\frac{3a-8}{12}$.

4. $\frac{2m+5n}{8m^2n^2}-\frac{3m+4n}{6mn^3}$.

5. $\frac{5a-7b}{27a}+\frac{a+6b}{36b}$.

6. $\frac{x-y}{xy}+\frac{y-2z}{2yz}+\frac{z-3x}{3zx}$.

7. $\frac{2(6n+5)}{11}-\frac{3(n+6)}{22}+\frac{4(5n-4)}{44}$.

8. $\frac{3a-2}{3a^3}+\frac{4a-7}{7a^2}-\frac{7a-3}{9a}$.

9. $\frac{8x+1}{7x}-\frac{10y-9}{14y}-\frac{9z+8}{21z}$.

10. $\frac{2a^2+3}{6a^2}-\frac{3a^3+1}{12a^3}-\frac{3a^4-2}{36a^4}$.

11. $\frac{4x-3}{5}+\frac{6x+5}{10}-\frac{5x+2}{15}-\frac{3x-10}{20}$.

12. $\frac{3m-2}{4}-\frac{7m-8}{6}+\frac{9m+4}{8}-\frac{10m+7}{9}$.

13. $\frac{2x+y}{8}-\frac{5x+4y}{16}-\frac{8x-3y}{24}+\frac{11x-2y}{32}$.

14. $\frac{3}{5m-2}+\frac{2}{2m+3}$.

15. $\frac{4}{3x-7}-\frac{1}{4x+5}$.

16. $\frac{m}{m+2}+\frac{2}{m-2}$.

17. $\frac{a+3}{a-3}+\frac{a-3}{a+3}$.

18. $\frac{2x}{2x+y}-\frac{y}{2x-y}$.

19. $\frac{5}{3a-9}-\frac{7}{5a-15}$.

20. $\frac{5x}{x-3}-\frac{4x^2+3x-1}{x^2+x-12}$.

21. $\frac{x+3y}{x-3y}-\frac{x-3y}{x+3y}$.

22. $\dfrac{a}{a^2+4a-60}-\dfrac{a}{a^2-4a-12}$.

23. $\dfrac{x}{2x-3}-\dfrac{x}{2x+3}-\dfrac{2x-6}{4x^2-9}$.

24. $\dfrac{a-n}{2a+2n}-\dfrac{3a-4n}{3a+3n}+\dfrac{3a-5n}{6a+6n}$.

25. $\dfrac{2x}{16x^2-8x+1}-\dfrac{2x}{16x^2-1}$.

26. $\dfrac{3x+2}{3x-2}-\dfrac{9x^2+4}{9x^2-4}$.

27. $\dfrac{x}{x+y}-\dfrac{x^2}{(x+y)^2}+\dfrac{x^3-xy^2}{(x+y)^3}$.

28. $\dfrac{a+b}{2a-2b}+\dfrac{a-b}{2a+2b}-\dfrac{2ab}{a^2-b^2}$.

29. $\dfrac{2a}{2a-1}-\dfrac{3a^2}{(2a-1)^2}-\dfrac{a^3}{(2a-1)^3}$.

30. $\dfrac{m-1}{m-2}-\dfrac{m+1}{m+2}+\dfrac{m-6}{m^2-4}$.

31. $\dfrac{x}{x+y}+\dfrac{x^2}{x^2-xy+y^2}-\dfrac{2x^3}{x^3+y^3}$.

32. $\dfrac{b}{(a+b)(b+c)}+\dfrac{c}{(b+c)(c+a)}-\dfrac{a}{(c+a)(a+b)}$.

33. $\dfrac{1}{a-3b}-\dfrac{3}{a+3b}+\dfrac{2a}{(a+3b)^2}$.

34. $\dfrac{x-3}{x+3}-\dfrac{x^3-27}{x^3+27}$.

35. $\dfrac{1}{a^2+4ab+4b^2}+\dfrac{1}{a^2-4b^2}-\dfrac{1}{a^2-2ab}$.

36. $\dfrac{5x+4}{x-4}-\dfrac{3x-2}{x+1}-\dfrac{2x^2+19x-8}{x^2-3x-4}$.

37. $\dfrac{2}{a+x}+\dfrac{3}{a-x}-\dfrac{5}{a+2x}$.

38. $\dfrac{1}{2x^2+5x+3}-\dfrac{1}{4x^2+8x+3}$.

39. $\dfrac{1}{x-1}-\dfrac{x}{x^2-1}+\dfrac{x^3}{x^3-1}.$ **40.** $\dfrac{a-1}{a+1}+\dfrac{a+1}{a-1}-\dfrac{a^2-1}{a^2+1}.$

41. $\dfrac{4n-1}{6n^2-17n+12}+\dfrac{3n+1}{10n^2-9n-9}.$

42. $\dfrac{a-4}{2a-3}-\dfrac{3a-1}{a+2}+\dfrac{5a^2-9a+11}{2a^2+a-6}.$

43. $\dfrac{x+4}{x^2-x-6}-\dfrac{x-2}{x^2-7x+12}+\dfrac{x+3}{x^2-2x-8}.$

44. $\dfrac{1}{m^2-mn+n^2}-\dfrac{1}{m^2+mn+n^2}-\dfrac{2mn}{m^4+m^2n^2+n^4}.$

147. In certain cases, the principles of §§ 137 and 138 enable us to change the form of a fraction to one which is more convenient for the purposes of addition or subtraction.

1. Simplify $\dfrac{3}{a-b}+\dfrac{2b+a}{b^2-a^2}.$

Changing the signs of the terms in the second denominator, at the same time changing the sign before the fraction (§ 137), we have

$$\frac{3}{a-b}-\frac{2b+a}{a^2-b^2}.$$

The L. C. D. is now a^2-b^2.

Then,
$$\frac{3}{a-b}-\frac{2b+a}{a^2-b^2}=\frac{3(a+b)-(2b+a)}{a^2-b^2}$$
$$=\frac{3a+3b-2b-a}{a^2-b^2}=\frac{2a+b}{a^2-b^2}.$$

2. Simplify $\dfrac{1}{(x-y)(x-z)}-\dfrac{1}{(y-x)(y-z)}-\dfrac{1}{(z-x)(z-y)}.$

By § 138, we change the sign of the factor $y-x$ in the second denominator, at the same time changing the sign before the fraction; and we change the signs of both factors of the third denominator.

The expression then becomes

$$\frac{1}{(x-y)(x-z)}+\frac{1}{(x-y)(y-z)}-\frac{1}{(x-z)(y-z)}.$$

The L. C. D. is now $(x-y)(x-z)(y-z)$; then the result

$$=\frac{(y-z)+(x-z)-(x-y)}{(x-y)(x-z)(y-z)}=\frac{y-z+x-z-x+y}{(x-y)(x-z)(y-z)}$$

$$=\frac{2y-2z}{(x-y)(x-z)(y-z)}=\frac{2(y-z)}{(x-y)(x-z)(y-z)}=\frac{2}{(x-y)(x-z)}.$$

EXERCISE 53

Simplify the following:

1. $\frac{4}{3a-3}-\frac{1}{2-2a}.$

2. $\frac{3x}{x^2-16}+\frac{2}{4-x}.$

3. $\frac{a+b}{a^2-3ab}-\frac{a-b}{3b^2-ab}.$

4. $\frac{5}{2m-1}+\frac{8m+6}{1-4m^2}.$

5. $\frac{a}{3+a}+\frac{a}{3-a}-\frac{2a^2}{a^2-9}.$

6. $\frac{4}{x^3-x}-\frac{3x}{1-x}-\frac{2}{x}.$

7. $\frac{1}{n+4}-\frac{1}{1-n}+\frac{n-6}{n^2+3n-4}.$

8. $\frac{3a}{a+2b}+\frac{2a}{2b-a}+\frac{8ab}{a^2-4b^2}.$

9. $\frac{6x^2-8x-32}{9x^3-16x}-\frac{2}{4-3x}-\frac{1}{x}.$

10. $\frac{x}{(x-y)(x-z)}+\frac{y}{(y-x)(y-z)}.$

11. $\frac{2a^2-4ab+b^2}{a^3-b^3}-\frac{3a}{a^2+ab+b^2}-\frac{1}{b-a}.$

12. $\frac{1}{x^2-5x+6}+\frac{1}{x^2-x-2}+\frac{1}{4-x^2}.$

13. $\frac{3m+1}{3m-1}+\frac{m-4}{5-2m}-\frac{3m^2-2m-4}{6m^2-17m+5}.$

14. $\frac{1}{(a-b)(a-c)}+\frac{1}{(b-c)(b-a)}+\frac{1}{(c-a)(c-b)}.$

148. Reduction of a Mixed Expression to a Fraction.

Ex. Reduce $2x-3-\frac{4x-5}{x+1}$ to a fractional form.

We may regard $2x-3$ as a fraction having the denominator 1, and use the rule of § 145; thus,

$$2x-3-\frac{4x-5}{x+1}=\frac{(2x-3)(x+1)-(4x-5)}{x+1}$$
$$=\frac{2x^2-x-3-4x+5}{x+1}=\frac{2x^2-5x+2}{x+1}.$$

EXERCISE 54

Reduce each of the following to a fractional form:

1. $x+3+\frac{x-5}{4x}$.

2. $2a-5-\frac{8a^2-9}{7a}$.

3. $3n+4+\frac{2}{5n+1}$.

4. $\frac{a-2m}{a+2m}+1$.

5. $1-\frac{5x+y}{5x-y}$.

6. $5-3n+\frac{25+9n^2}{5+3n}$.

7. $2-\frac{2a-7b}{3a+4b}$.

8. $x^2+2xy+4y^2+\frac{16y^3}{x-2y}$.

9. $a-4b-\frac{a^2+16b^2}{a-4b}$.

10. $2x^2-5x+\frac{2x(x+17)}{4x+9}$.

11. $3a^2+8-\frac{5a(4a-1)}{7a-2}$.

12. $\frac{8x^2+y^2}{3x+5y}-(2x+y)$.

13. $\frac{a}{a+b}+\frac{b}{a-b}-1$.

14. $m^3-m^2n+mn^2-n^3-\frac{2n^4}{m+n}$.

15. $x-3+\frac{x^3+27}{x^2+3x+9}$.

16. $\frac{(x-2)^2}{(x+2)^2}-\frac{2(x-2)}{x+2}+1$.

17. $\frac{(a^2+4)(a^2-8)}{a^2-2a+4}+a^2+2a+4$.

18. $2x+5y-\frac{2x^3+25y^3}{x^2-3xy+5y^2}$.

MULTIPLICATION OF FRACTIONS

149. Required the product of $\frac{a}{b}$ and $\frac{c}{d}$.

Let $$\frac{a}{b}\times\frac{c}{d}=x. \qquad (1)$$

Multiplying both members by $b \times d$ (Ax. 7, § 9),

$$\frac{a}{b} \times \frac{c}{d} \times b \times d = x \times b \times d, \text{ or } \left(\frac{a}{b} \times b\right) \times \left(\frac{c}{d} \times d\right) = x \times b \times d;$$

for the factors of a product may be written in any order.

Now since the product of the quotient and the divisor gives the dividend (§ 67), we have

$$\frac{a}{b} \times b = a, \text{ and } \frac{c}{d} \times d = c.$$

Whence, $(a) \times (c) = x \times b \times d.$

Dividing both members by $b \times d$ (Ax. 8, § 9),

$$\frac{a \times c}{b \times d} = x. \qquad (2)$$

From (1) and (2), $\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}.$ (Ax. 4, § 9)

Then, to multiply fractions, *multiply the numerators together for the numerator of the product, and the denominators for its denominator.*

150. Since c may be regarded as a fraction having the denominator 1, we have, by § 149,

$$\frac{a}{b} \times c = \frac{a}{b} \times \frac{c}{1} = \frac{ac}{b}.$$

Dividing both numerator and denominator by c (§ 136),

$$\frac{a}{b} \times c = \frac{a}{b \div c}.$$

Then, to multiply a fraction by a rational and integral expression, *if possible, divide the denominator of the fraction by the expression; otherwise, multiply the numerator by the expression.*

151. Common factors in the numerators and denominators should be cancelled before performing the multiplication.

Mixed expressions should be expressed in a fractional form (§ 148) before applying the rules.

1. Multiply $\dfrac{10\,a^3y}{9\,bx^2}$ by $\dfrac{3\,b^4x^3}{4\,a^3y^2}$.

$$\frac{10\,a^3y}{9\,bx^2}\times\frac{3\,b^4x^3}{4\,a^3y^2}=\frac{2\times 5\times 3\times a^3b^4x^3y}{3^2\times 2^2\times a^3bx^2y^2}=\frac{5\,b^3x}{6\,y}.$$

The factors cancelled are 2, 3, a^3, b, x^2, and y.

2. Multiply together $\dfrac{x^2+2\,x}{x^2+x-6}$, $2-\dfrac{x-4}{x-3}$, and $\dfrac{x^2-9}{x^2-4}$.

$$\frac{x^2+2\,x}{x^2+x-6}\times\left(2-\frac{x-4}{x-3}\right)\times\frac{x^2-9}{x^2-4}$$

$$=\frac{x^2+2\,x}{x^2+x-6}\times\frac{2\,x-6-x+4}{x-3}\times\frac{x^2-9}{x^2-4}$$

$$=\frac{x(x+2)}{(x+3)(x-2)}\times\frac{x-2}{x-3}\times\frac{(x+3)(x-3)}{(x+2)(x-2)}=\frac{x}{x-2}.$$

The factors cancelled are $x+2$, $x-2$, $x+3$, and $x-3$.

3. Multiply $\dfrac{a^2+b^2}{a^2-b^2}$ by $a-b$.

Dividing the denominator by $a-b$, $\dfrac{a^2+b^2}{a^2-b^2}\times(a-b)=\dfrac{a^2+b^2}{a+b}$.

4. Multiply $\dfrac{m}{m-n}$ by $m+n$.

Multiplying the numerator by $m+n$, $\dfrac{m}{m-n}\times(m+n)=\dfrac{m^2+mn}{m-n}$.

EXERCISE 55

Simplify the following:

1. $\dfrac{8\,am^2}{27\,b^4n^5}\times 9\,bn^5$.

2. $\dfrac{21\,a^3b^2}{8\,cd^6}\times\dfrac{4\,c^5d^6}{35\,a^3b^5}$.

3. $\dfrac{5\,a^2}{3\,b^2}\times\dfrac{9\,b^3}{10\,c^3}\times\dfrac{7\,c^4}{6\,a^4}$.

4. $\dfrac{3\,x^5}{10\,y^3}\times\dfrac{15\,y^7}{7\,z}\times\dfrac{28\,z^3}{9\,x^2}$.

5. $\dfrac{14\,b^3c}{15\,a^6}\times\dfrac{5\,c^3a}{12\,b^5}\times\dfrac{6\,a^2b}{7\,c^4}$.

6. $\dfrac{28\,m^7}{25\,n^5x^3}\times\dfrac{15\,n^6}{14\,m^5x^2}\times\dfrac{5\,x^8}{21\,m^3n^4}$

7. $\dfrac{3a+b}{a-2b}\times(2a-b)$.

8. $\dfrac{x-2}{9x^2-16}\times(3x+4)$.

9. $\dfrac{n^2-36}{4n^2}\times\dfrac{7n^2}{n^2+n-42}$.

10. $\dfrac{a^2-2a-35}{2a^3-3a^2}\times\dfrac{4a^3-9a}{a-7}$.

11. $\dfrac{x^2+9xy+18y^2}{x^2-9xy+20y^2}\times\dfrac{xy^2-4y^3}{x^2+6xy+9y^2}$.

12. $\dfrac{2a^2-5a}{a^3-27}\times\dfrac{a^2+3a+9}{4a^2-20a+25}$.

13. $\dfrac{5x+2}{2x^2+x-10}\times(x-2)$.

14. $\dfrac{4m^2+8m+3}{2m^2-5m+3}\times\dfrac{6m^2-9m}{4m^2-1}$.

15. $\dfrac{x^2+mx+nx+mn}{x^2-mx-nx+mn}\times\dfrac{x^2-m^2}{x^2-n^2}$.

16. $\dfrac{a^2-2ab+b^2-c^2}{a^2+2ab+b^2-c^2}\times\dfrac{a+b-c}{a-b+c}$.

17. $\dfrac{16x-4}{5x+5}\times\dfrac{20x+5}{6x+6}\times\dfrac{x^2+2x+1}{16x^2-1}$.

18. $\dfrac{a^2-11a+30}{a^3-6a^2+9a}\times\dfrac{a^2-3a}{a^2-25}\times\dfrac{a^2-9}{a^2+3a-54}$.

19. $\left(3a+\dfrac{10a^2-9a-25}{2a+3}\right)\left(2a-\dfrac{4a^2-22a-9}{4a-5}\right)$.

20. $\dfrac{x^3+8y^3}{x^3-8y^3}\times\dfrac{x-2y}{x+2y}\times\left(1+\dfrac{4xy}{x^2-2xy+4y^2}\right)$.

21. $\dfrac{9x^2+12ax+4a^2}{x^4-a^4}\times\dfrac{x^2+ax}{3x+2a}\times\left(x-\dfrac{2x^2+2ax-a^2}{3x+2a}\right)$.

22. $\dfrac{2n^2-n-3}{n^4-8n^2+16}\times\dfrac{n^2+4n+4}{n^2+n}\times\dfrac{n^2-n-2}{2n^2-3n}$.

DIVISION OF FRACTIONS

152. Required the quotient of $\dfrac{a}{b}$ divided by $\dfrac{c}{d}$.

Let $$\frac{a}{b} \div \frac{c}{d} = x. \qquad (1)$$

Then since the dividend is the product of the divisor and quotient (§ 67), we have

$$\frac{a}{b} = \frac{c}{d} \times x.$$

Multiplying both members by $\frac{d}{c}$ (Ax. 7, § 9),

$$\frac{a}{b} \times \frac{d}{c} = \frac{c}{d} \times x \times \frac{d}{c} = x. \qquad (2)$$

From (1) and (2), $$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}. \qquad \text{(Ax. 4, § 9)}$$

Then, *to divide one fraction by another, multiply the dividend by the divisor inverted.*

153. Since c may be regarded as a fraction having the denominator 1, we have, by § 152,

$$\frac{a}{b} \div c = \frac{a}{b} \times \frac{1}{c} = \frac{a}{bc}.$$

Dividing both numerator and denominator by c (§ 136),

$$\frac{a}{b} \div c = \frac{a \div c}{b}.$$

Therefore, to divide a fraction by a rational and integral expression:

If possible, divide the numerator of the fraction by the expression; otherwise, multiply the denominator by the expression.

154. Mixed expressions should be expressed in a fractional form (§ 148) before applying the rules.

1. Divide $\frac{6a^2b}{5x^3y^4}$ by $\frac{9a^2b^3}{10x^2y^7}$.

We have, $$\frac{6a^2b}{5x^3y^4} \div \frac{9a^2b^3}{10x^2y^7} = \frac{6a^2b}{5x^3y^4} \times \frac{10x^2y^7}{9a^2b^3} = \frac{4y^3}{3b^2x}.$$

2. Divide $2-\dfrac{2x-3}{x+1}$ by $3-\dfrac{3x^2-13}{x^2-1}$.

$$\left(2-\frac{2x-3}{x+1}\right)\div\left(3-\frac{3x^2-13}{x^2-1}\right)$$

$$=\frac{2x+2-2x+3}{x+1}\div\frac{3x^2-3-3x^2+13}{x^2-1}$$

$$=\frac{5}{x+1}\times\frac{x^2-1}{10}=\frac{5(x+1)(x-1)}{2\times5\times(x+1)}=\frac{x-1}{2}.$$

3. Divide $\dfrac{m^3-n^3}{m^2+n^2}$ by $m-n$.

Dividing the numerator by $m-n$, $\dfrac{m^3-n^3}{m^2+n^2}\div(m-n)=\dfrac{m^2+mn+n^2}{m^2+n^2}.$

4. Divide $\dfrac{a^2+b^2}{a-b}$ by $a+b$.

Multiplying the denominator by $a+b$, $\dfrac{a^2+b^2}{a-b}\div(a+b)=\dfrac{a^2+b^2}{a^2-b^2}.$

If the numerator and denominator of the divisor are exactly contained in the numerator and denominator, respectively, of the dividend, it follows from § 149 that *the numerator of the quotient may be obtained by dividing the numerator of the dividend by the numerator of the divisor; and the denominator of the quotient by dividing the denominator of the dividend by the denominator of the divisor.*

5. Divide $\dfrac{9x^2-4y^2}{x^2-y^2}$ by $\dfrac{3x+2y}{x-y}$.

We have, $$\frac{9x^2-4y^2}{x^2-y^2}\div\frac{3x+2y}{x-y}=\frac{3x-2y}{x+y}.$$

EXERCISE 56

Simplify the following:

1. $\dfrac{45x^6m^2}{4y^5n}\div 9x^3m^4.$

2. $\dfrac{12a^6b^2}{55c^3d^5}\div\dfrac{9a^4b^3}{22c^7d^5}.$

3. $\frac{4a^2-25}{a-3} \div (2a-5)$.

4. $\frac{3x-2y}{4x+y} \div (x+3y)$.

5. $\frac{n^2-3n-40}{4n} \div \frac{n^2+4n-5}{5n}$.

6. $\frac{a^2-ab-2b^2}{a^3-9ab^2} \div \frac{a-2b}{a+3b}$.

7. $\frac{9x^2-4y^2}{16x^2-25y^2} \div \frac{9x^2+6xy}{8xy-10y^2}$.

8. $\frac{a^2-10a+16}{a^2+6a+9} \div \frac{a^2-4}{a+3}$.

9. $\left(\frac{x^3}{4y}+\frac{x^2}{3}\right) \div \left(\frac{x^3}{5y}-\frac{x^2}{2}\right)$.

10. $\frac{x^2-3xy}{x^3-y^3} \div \frac{x^2-10xy+21y^2}{x^2+xy+y^2}$.

11. $\frac{4x^2+12x+5}{4x-3} \div (2x+1)$.

12. $\frac{8n^3+1}{2n^2+4n} \div \frac{4n^2-2n+1}{n^2+4n+4}$.

13. $\left(2-\frac{2+8x-3x^2}{9-x^2}\right) \div \left(6-\frac{14+7x}{3+x}\right)$.

14. $(x^2-y^2+2yz-z^2) \div \frac{x-y+z}{x+y+z}$.

15. $\frac{m^3+2m^2+m+2}{m^3-m^2-m+1} \div \frac{m^4+3m^2+2}{m^4-2m^2+1}$.

16. $\frac{2a^2-ab-3b^2}{9a^2-25b^2} \div \frac{3a^2+ab-2b^2}{9a^2-30ab+25b^2}$.

COMPLEX FRACTIONS

155. A **Complex Fraction** is a fraction having one or more fractions in either or both of its terms.

It is simply a case in division of fractions; its numerator being the dividend, and its denominator the divisor.

1. Simplify $\dfrac{a}{b-\frac{c}{d}}$.

$$\frac{a}{b-\frac{c}{d}}=\frac{a}{\frac{bd-c}{d}}=a\times\frac{d}{bd-c}\ (\S\ 152)=\frac{ad}{bd-c}.$$

It is often advantageous to simplify a complex fraction by multiplying its numerator and denominator by the L. C. M. of their denominators (§ 136).

2. Simplify $\dfrac{\dfrac{a}{a-b}-\dfrac{a}{a+b}}{\dfrac{b}{a-b}+\dfrac{a}{a+b}}$.

The L. C. M. of $a+b$ and $a-b$ is $(a+b)(a-b)$.
Multiplying both terms by $(a+b)(a-b)$, we have

$$\frac{\dfrac{a}{a-b}-\dfrac{a}{a+b}}{\dfrac{b}{a-b}+\dfrac{a}{a+b}}=\frac{a(a+b)-a(a-b)}{b(a+b)+a(a-b)}=\frac{a^2+ab-a^2+ab}{ab+b^2+a^2-ab}=\frac{2ab}{a^2+b^2}.$$

3. Simplify $\dfrac{1}{1+\dfrac{1}{1+\dfrac{1}{x}}}$.

$$\frac{1}{1+\dfrac{1}{1+\dfrac{1}{x}}}=\frac{1}{1+\dfrac{x}{x+1}}=\frac{x+1}{x+1+x}=\frac{x+1}{2x+1}.$$

In examples like the above, it is best to begin by simplifying the *lowest* complex fraction.

Thus, we first multiply both terms of $\dfrac{1}{1+\dfrac{1}{x}}$ by x, giving $\dfrac{x}{x+1}$; and then multiply both terms of $\dfrac{1}{1+\dfrac{x}{x+1}}$ by $x+1$, giving $\dfrac{x+1}{x+1+x}$.

EXERCISE 57

Simplify the following:

1. $\dfrac{\dfrac{2}{x}+\dfrac{5}{y}}{\dfrac{3}{x}-\dfrac{4}{y}}$. **2.** $\dfrac{1-\dfrac{2}{3a}}{a-\dfrac{4}{9a}}$. **3.** $\dfrac{m+\dfrac{8}{m^2}}{1+\dfrac{2}{m}}$.

4. $\dfrac{\dfrac{x-y}{x}-\dfrac{x+y}{y}}{\dfrac{x+y}{x}+\dfrac{x-y}{y}}$. **5.** $\dfrac{\dfrac{a+b}{b}-\dfrac{b}{a+b}}{\dfrac{2}{a}+\dfrac{1}{b}}$.

6. $$\frac{\frac{3a}{4b}+2+\frac{4b}{3a}}{\frac{4}{a}+\frac{3}{b}}.$$

7. $$\frac{\frac{x}{y}-4-\frac{21y}{x}}{\frac{x}{y}+1-\frac{6y}{x}}.$$

8. $$\frac{\frac{x^2y^2}{1-x^2y^2}+\frac{1+x^2y^2}{x^2y^2}}{\frac{xy}{1+xy}+\frac{1-xy}{xy}}.$$

9. $$\frac{\frac{1}{1-x}-\frac{1}{1+x}}{\frac{1}{1-x^2}-\frac{1}{1+x^2}}.$$

10. $$\frac{\left(1+\frac{1}{x}\right)\left(1+\frac{1}{x^2}\right)}{x+2+\frac{1}{x}}.$$

11. $$1-\frac{5}{1-\frac{5}{1-\frac{5}{1+a}}}.$$

12. $$\frac{x-7+\frac{10}{x}}{2x-9-\frac{5}{x}}.$$

13. $$\frac{2}{5a-\frac{4a-1}{1-\frac{2a+5}{3a-2}}}.$$

14. $$\frac{\frac{27a^2}{b^2}-\frac{b}{a}}{\frac{9a}{b}+3+\frac{b}{a}}.$$

15. $$\frac{6+\frac{x^2+5y^2}{4x^2-y^2}}{3-\frac{x-4y}{2x-y}}.$$

16. $$\frac{\frac{x+4}{5x-1}-\frac{x-4}{5x+1}}{\frac{x+4}{5x-1}+\frac{x-4}{5x+1}}.$$

17. $$\frac{\frac{x^2-y^2-z^2}{2yz}-1}{\frac{x^2+y^2-z^2}{2xy}-1}.$$

18. $$\frac{\frac{3a}{(a+2)^2}+\frac{a-2}{a+2}}{\frac{2a^2+2a-1}{a^2-4}-\frac{a}{a-2}}.$$

19. $$\frac{\frac{3x^2+y^2}{(x-y)^2}-\frac{x+y}{x-y}}{\frac{x}{x+y}+\frac{y}{x-y}}.$$

20. $$\frac{x^2-\frac{x^2y+y^3}{x+y}}{x+\frac{y^2}{x+y}}.$$

21. $$\frac{\frac{x+1}{x-1}-\frac{x^2+1}{x^2-1}}{\frac{x-1}{x+1}-\frac{x^2-1}{x^2+1}}.$$

22. $$\dfrac{\dfrac{a-x}{a+x}-\dfrac{a^3-x^3}{a^3+x^3}}{\dfrac{a-x}{a+x}+\dfrac{a^3-x^3}{a^3+x^3}}.$$

MISCELLANEOUS AND REVIEW EXAMPLES

EXERCISE 57 a

Reduce each of the following to a mixed expression:

1. $\dfrac{9x^4-2x^3-20}{12x^3}$.
2. $\dfrac{m^5+n^5}{m-n}$.
3. $\dfrac{12a^3-3a^2-22a+8}{3a^2-5}$.

Simplify the following:

4. $\dfrac{3x-y}{x^2-2y^2}\times(2x+y)$.
5. $\dfrac{a^3+64b^3}{a^3-64b^3}\times(a-4b)$.
6. $\dfrac{a^4-n^4}{a^4+n^4}\div(a+n)$.
7. $\dfrac{2mx}{4x+5m}\div(2m-3x)$.
8. $\dfrac{x^2-5x-84}{27x^3-8}\div\dfrac{x+7}{3x-2}$.
9. $3x+4-\dfrac{5x+7}{2x-3}$.
10. $\dfrac{a^2+b^2}{a-b}-(a-b)$.
11. $\dfrac{3}{5x^2}-\dfrac{2}{15xy}+\dfrac{1}{6y^2}$.
12. $\dfrac{x^4+2x^3-8x-16}{x^4-2x^3+8x-16}$.
13. $\dfrac{(a-b)^2-(c-d)^2}{(b+d)^2-(a+c)^2}$.
14. $\dfrac{\dfrac{1}{2x}+\dfrac{1}{3y}}{4x^2-9y^2}+\dfrac{\dfrac{1}{3x}-\dfrac{1}{2y}}{9x^2-4y^2}$.
15. $\dfrac{a^2bd-ab^2c-acd^2+bc^2d}{b^2cd+abc^2-abd^2-a^2cd}$.
16. $\left(\dfrac{a+2}{a}+\dfrac{2}{a-3}\right)\left(\dfrac{a}{a-2}-\dfrac{3}{a+3}\right)$.
17. $\dfrac{x^3-2x^2+2x-1}{x^4+x^2+1}$.
18. $\dfrac{1}{a-b}+\dfrac{4}{b-a}-\dfrac{8}{a+b}-\dfrac{11a-5b}{b^2-a^2}$.
19. $\dfrac{27x^3+1}{25x^2-4}\div\dfrac{15x^2-x-2}{25x^2-20x+4}$.

20. $\left(2-\dfrac{x^2+4x-21}{x^2+2x-8}\right)\div\left(\dfrac{x+1}{x-2}+\dfrac{x-3}{x+4}\right).$

21. $\dfrac{2c-d}{ac+2ad+2bc+4bd}-\dfrac{c+2d}{2ac-ad+4bc-2bd}.$

22. $\left(2x-1+\dfrac{6x-11}{x+4}\right)\div\left(x+3-\dfrac{3x+17}{x+4}\right).$

23. $\dfrac{a+3}{4(a^2+3a+2)}+\dfrac{a+2}{a^2+4a+3}-\dfrac{a+1}{4(a^2+5a+6)}.$

24. $\dfrac{2ac-6ad-bc+3bd}{3ac+ad-6bc-2bd}\times\dfrac{a^2-7ab+10b^2}{10a^2-3ab-b^2}.$

25. $\dfrac{y-z}{x^2-(y-z)^2}-\dfrac{z-x}{(x-z)^2-y^2}.$

26. $\dfrac{(a+2b+3c)^2-(2a-3b-c)^2}{(3a+b-2c)^2-(2a+6b+2c)^2}.$

27. $\dfrac{1}{x+3}-\dfrac{1}{x-3}-\dfrac{2}{x+4}+\dfrac{2}{x-4}.$

(First combine the first two fractions, then the last two, and then add these results.)

28. $\dfrac{3}{2n+1}+\dfrac{3}{2n-1}-\dfrac{5n^2}{8n^3+1}-\dfrac{5n^2}{8n^3-1}.$

29. $\left[\left(\dfrac{1}{y-z}-\dfrac{1}{x}\right)\div\left(\dfrac{1}{y+z}-\dfrac{1}{x}\right)\right]\div\dfrac{x^2-y^2-z^2+2yz}{x^2-y^2-z^2-2yz}.$

30. $\dfrac{b-c}{(a-b)(a-c)}+\dfrac{c-a}{(b-c)(b-a)}-\dfrac{a-b}{(c-a)(c-b)}.$

31. $\dfrac{(2x^2+5x-2)^2-25}{(3x^2-4x-3)^2-16}.$

32. $\dfrac{a}{a+3}+\dfrac{1}{a-3}-\dfrac{3}{a^2-9}-\dfrac{a^2+2a}{a^2+9}.$

(First add the first two fractions, to the result add the third fraction, and to this result add the last fraction.)

33. $\dfrac{3a}{a+b}+\dfrac{3a}{a-b}+\dfrac{6a^2}{a^2+b^2}+\dfrac{12a^4}{a^4+b^4}.$

34. $\dfrac{3}{2(a-1)}-\dfrac{1}{2(a+1)}+\dfrac{a-2}{a^2+1}-\dfrac{2a^3+4}{a^4-1}$.

35. $\dfrac{1}{2x^2+3x-2}-\dfrac{1}{3x^2+5x-2}+\dfrac{1}{1+x-6x^2}$.

36. $\dfrac{\dfrac{2}{x+y}-\dfrac{1}{x}}{y-\dfrac{xy}{2x+y}}-\dfrac{\dfrac{1}{y}-\dfrac{2}{x+y}}{x-\dfrac{xy}{x+2y}}$.

37. $\dfrac{x^3-2x^2-4x+8}{x^4+3x^3-27x-81}\div\left(\dfrac{x^2+9x+14}{x^2+6x+9}\times\dfrac{x^2-4x+4}{x^2-9}\right)$.

38. $\dfrac{6a^2-a-2}{4a^2-16a+15}\times\dfrac{8a^2-18a-5}{12a^2-5a-2}\times\dfrac{4a^2-9}{4a^2+8a+3}$.

39. $\dfrac{\left(\dfrac{x+1}{x-1}\right)^2-2+\left(\dfrac{x-1}{x+1}\right)^2}{\left(\dfrac{x+1}{x-1}\right)^2-\left(\dfrac{x-1}{x+1}\right)^2}$.

40. $\dfrac{x+2}{x-2}+\dfrac{x+3}{x-3}+\dfrac{x+4}{x-4}-\dfrac{3x^3-10x^2-21x+66}{(x-2)(x-3)(x-4)}$.

XI. FRACTIONAL AND LITERAL LINEAR EQUATIONS

SOLUTION OF FRACTIONAL LINEAR EQUATIONS

156. If a fraction whose numerator is a polynomial is preceded by a $-$ sign, it is convenient, on clearing of fractions, to enclose the numerator in parentheses, as shown in Ex. 1.

If this is not done, care must be taken to *change the sign of each term of the numerator* when the denominator is removed.

1. Solve the equation $\frac{3x-1}{4}-\frac{4x-5}{5}=4+\frac{7x+5}{10}$.

The L. C. M. of 4, 5, and 10 is 20.
Multiplying each term by 20, we have

$$15x-5-(16x-20)=80+14x+10.$$

Whence, $\quad 15x-5-16x+20=80+14x+10.$

Transposing, $\quad 15x-16x-14x=80+10+5-20.$

Uniting terms, $\quad -15x=75.$

Dividing by -15, $\quad x=-5.$

2. Solve the equation $\frac{2}{x-2}-\frac{5}{x+2}-\frac{2}{x^2-4}=0$.

The L. C. M. of $x-2$, $x+2$, and x^2-4 is x^2-4.
Multiplying each term by x^2-4, we have

$$2(x+2)-5(x-2)-2=0.$$

Or, $\quad 2x+4-5x+10-2=0.$

Transposing, and uniting terms, $-3x=-12$, and $x=4$.

If the denominators are partly monomial and partly polynomial, it is often advantageous to clear of fractions at first partially; multiplying each term of the equation by the L. C. M. of the *monomial* denominators.

3. Solve the equation $\frac{6x+1}{15}-\frac{2x-4}{7x-16}=\frac{2x-1}{5}$.

Multiplying each term by 15, the L. C. M. of 15 and 5,

$$6x+1-\frac{30x-60}{7x-16}=6x-3.$$

Transposing, and uniting terms, $4=\frac{30x-60}{7x-16}$.

Clearing of fractions, $28x-64=30x-60$.

Then, $-2x=4$, and $x=-2$.

EXERCISE 58

In Exs. 5, 11, 22, and 32, of the following set, other letters than x are used to represent unknown numbers.

This is done repeatedly in the later portions of the work.

Solve the following equations:

1. $\frac{1}{2}-\frac{4}{9x}=\frac{4}{9}-\frac{1}{6x}$.

2. $\frac{1}{5x}-\frac{3}{10x}-\frac{2}{15x}=-\frac{7}{12}$.

3. $\frac{4}{3x}-\frac{1}{12x}-\frac{5}{8x}+\frac{7}{24x}=-\frac{11}{8}$.

4. $4x+\frac{6x+7}{5}=-\frac{5x}{2}$.

5. $\frac{6y}{5}+3-\frac{2y+3}{15}=2y$.

6. $\frac{7x}{4}-\frac{8x-9}{7}-2=\frac{x}{14}$.

7. $x-\frac{4x+7}{3}+\frac{5x+9}{8}=-2$.

8. $\frac{3x+2}{5}-\frac{5x-6}{8}=\frac{5}{4}$.

9. $\frac{8x-11}{9}-\frac{7x+4}{12}-\frac{3x-8}{8}=0$.

10. $\frac{5(x-1)}{6}-\frac{2(x+2)}{3}=4-\frac{5x-15}{4}$.

11. $\frac{11p+12}{18}-\frac{4p-6}{9}+\frac{5p-9}{4}=-3$.

12. $\frac{8x-1}{3}-\frac{11x-7}{5}-\frac{13x+3}{10}=\frac{14x+38}{15}$.

13. $\frac{5x+4}{3}-\frac{16x+5}{9}=\frac{10x-9}{5}-\frac{4(3x-2)}{15}.$

14. $\frac{3(x+7)}{7x}-\frac{7x+10}{3x}=\frac{4x-7}{6}-\frac{2(7x-1)}{21}.$

15. $\frac{(3x-4)(3x+1)}{2}-\frac{(8x-11)(x+1)}{4}=\frac{(5x-1)(4x-3)}{8}.$

16. $\frac{5}{4x-3}-\frac{8}{7x-3}=0.$

17. $\frac{6x+1}{9x-5}=\frac{2x+3}{3x-2}.$

18. $\frac{21x^2+7x+11}{7x^2-4x-9}=3.$

19. $\frac{2x+7}{x^2-4}=\frac{10x-3}{5x(x+2)}.$

20. $\frac{8x+57}{12}=\frac{2x-15}{x+8}+\frac{2x+16}{3}.$

21. $\frac{12x-5}{21}-\frac{3x+4}{3(3x+1)}=\frac{4x-5}{7}.$

22. $\frac{3n-1}{n-5}-\frac{5n+4}{n+8}=-2.$

23. $\frac{6x^2+23}{(2x-3)^2}+\frac{x-1}{2x-3}=2.$

24. $\frac{2x-5}{9}=x-1-\frac{7x^2+11}{3(3x+4)}.$

25. $\frac{5x-4}{7}-\frac{2x-7}{14}=\frac{4x^2-x}{7x-2}.$

26. $\frac{2}{2x+1}-\frac{1}{3x+2}+\frac{7}{6x^2+7x+2}=0.$

27. $\frac{1}{6x-24}-\frac{2}{6x+3}+\frac{1}{6x-4}=0.$

28. $\frac{4x^2+3x+2}{4x+3}=\frac{6x^2-5x-4}{6x-5}.$

29. $\frac{8x^2+4}{16x^2-25}=\frac{5x}{5+4x}+\frac{3x}{5-4x}.$

30. $\frac{7x}{x+3}-\frac{5x}{1-x}=\frac{12(x^2-1)}{x^2+2x-3}.$

31. $\frac{2x^2-x+3}{3x+2}-\frac{2x^2+3x-1}{3x-2}=\frac{-20x^2-6x+3}{9x^2-4}.$

32. $\dfrac{t-3}{t+1}+\dfrac{t+4}{t-2}=\dfrac{8t+18}{t^2-t-2}+2.$

33. $\dfrac{3x-5}{2}-\dfrac{4x+2}{3x+2}=\dfrac{15x-1}{10}+\dfrac{7}{5}.$

34. $\dfrac{x+1}{x+2}-\dfrac{x+2}{x+3}=\dfrac{x+5}{x+6}-\dfrac{x+6}{x+7}.$

(First add the fractions in the first member; then the fractions in the second member.)

35. $\dfrac{x}{x-2}-\dfrac{x+1}{x-1}=\dfrac{x-6}{x-4}-\dfrac{x-7}{x-5}.$

36. $\dfrac{4x+7}{5}-\dfrac{8x+4}{15}-\dfrac{12x+1}{45}=\dfrac{5x-1}{9(5x+2)}.$

37. $\dfrac{x^2+3}{2(x^3-8)}+\dfrac{1}{6(x-2)}=\dfrac{2x-1}{3(x^2+2x+4)}.$

38. $\dfrac{2x-1}{x-2}=\dfrac{1}{2}+\dfrac{2x+3}{3x+10}+\dfrac{5x^2+30x}{2(x-2)(3x+10)}.$

39. $\dfrac{2x+1}{3x-5}-\dfrac{5x-6}{2x+7}=2-\dfrac{23x^2-10}{6x^2+11x-35}.$

157. Solution of Special Forms of Fractional Equations.

1. Solve the equation $\dfrac{2x-1}{2x-3}+\dfrac{x^2-x}{x^2+4}=2.$

We divide each numerator by its corresponding denominator; then

$$1+\frac{2}{2x-3}+1-\frac{x+4}{x^2+4}=2, \text{ or } \frac{2}{2x-3}-\frac{x+4}{x^2+4}=0.$$

Clearing of fractions, $2x^2+8-(2x^2+5x-12)=0.$

Then, $\quad 2x^2+8-2x^2-5x+12=0$; whence, $x=4$.

We reject a solution which does not satisfy the given equation

2. Solve the equation $\dfrac{1}{x-3}+\dfrac{1}{x-2}=\dfrac{3x-7}{x^2-5x+6}.$

Multiplying both members by $(x-3)(x-2)$, or x^2-5x+6,

$$x-2+x-3=3x-7.$$

Transposing, and uniting terms, $-x=-2$, or $x=2$.

If we substitute 2 for x, the fraction $\frac{1}{x-2}$ becomes $\frac{1}{0}$.

Since division by 0 is impossible, the solution $x=2$ does not satisfy the given equation, and we reject it; the equation has no solution.

3. Solve the equation $\frac{3}{x+10}+\frac{4}{x+6}=\frac{2}{x+8}+\frac{5}{x+9}$.

Adding the fractions in each member, we have

$$\frac{7x+58}{(x+10)(x+6)}=\frac{7x+58}{(x+8)(x+9)}.$$

Clearing of fractions, and transposing all terms to the first member,

$$(7x+58)(x+8)(x+9)-(7x+58)(x+10)(x+6)=0. \qquad (1)$$

Factoring, $\quad(7x+58)[(x+8)(x+9)-(x+10)(x+6)]=0.$

Expanding, $\quad(7x+58)(x^2+17x+72-x^2-16x-60)=0.$

Or, $\quad(7x+58)(x+12)=0.$

This equation may be solved by the method of § 125.

Placing $7x+58=0$, we have $x=-\frac{58}{7}$.

Placing $x+12=0$, we have $x=-12$.

158. If we should solve equation (1), in Ex. 3 of § 157, by dividing both members by $7x+58$, we should have

$$(x+8)(x+9)-(x+10)(x+6)=0.$$

Then, $\quad x^2+17x+72-x^2-16x-60=0$, or $x=-12$.

In this way, the solution $x=-\frac{58}{7}$ is lost.

It follows from this that it is *never allowable to divide both members of an equation by any expression which involves the unknown numbers, unless the expression be placed equal to* 0 *and the root preserved, for in this way solutions are lost.*

EXERCISE 59

Solve the following equations:

1. $\frac{2x+7}{2x+1}+\frac{2x-3}{x-2}=3.$

2. $\frac{4x+11}{x^2+x-20}=\frac{1}{x+5}-\frac{1}{x-4}.$

3. $\frac{8}{x+3}-\frac{3}{x-7}=\frac{10}{x+9}-\frac{5}{x-2}.$

4. $\frac{x+3}{x+2}+\frac{x+4}{x+3}+\frac{x+2}{x+4}=3.$

5. $\frac{3}{x+9}+\frac{2}{x+4}=\frac{1}{x+3}+\frac{4}{x+18}.$

6. $\frac{2x+3}{2x-3}-\frac{2x-3}{2x+3}-\frac{36}{4x^2-9}=0.$

7. $\frac{2x+5}{x+7}-\frac{3x^2+24x+19}{x^2+8x+7}=-1.$

8. $\frac{x^2-2x+5}{x^2-2x-3}+\frac{x^2+3x-7}{x^2+3x+1}=2.$

9. $\frac{5}{2x-1}-\frac{1}{6x+5}=\frac{10}{3x-4}-\frac{4}{4x+1}.$

SOLUTION OF LITERAL LINEAR EQUATIONS

159. A Literal Equation is one in which some or all of the known numbers are represented by letters; as,

$$2x+a=b^2+10.$$

Ex. Solve the equation $\frac{x}{x-a}-\frac{x+2b}{x+a}=\frac{a^2+b^2}{x^2-a^2}.$

Multiplying each term by x^2-a^2,

$$x(x+a)-(x+2b)(x-a)=a^2+b^2,$$

or, $$x^2+ax-(x^2+2bx-ax-2ab)=a^2+b^2,$$

or, $$x^2+ax-x^2-2bx+ax+2ab=a^2+b^2,$$

or, $$2ax-2bx=a^2-2ab+b^2.$$

Factoring both members, $$2x(a-b)=(a-b)^2.$$

Dividing by $2(a-b)$, $$x=\frac{(a-b)^2}{2(a-b)}=\frac{a-b}{2}.$$

In solving fractional literal equations, we must reject any solution which does not satisfy the given equation. Compare Ex. 2, § 157.

EXERCISE 60

Solve the following equations:

1. $(ax-b)(bx+a)=b(ax^2-b).$

2. $(x-2a-b)^2=(x+a+2b)^2.$

3. $\frac{3x}{2x+n}+\frac{x-2n}{2x}=2.$

4. $\frac{4x+3a}{4x-3a}-\frac{6a+5b}{6a-5b}=0.$

5. $\dfrac{x}{ab}+\dfrac{x}{bc}+\dfrac{x}{ca}=a+b+c.$

6. $\dfrac{x(a+4b)-b^2}{a^2-b^2}+\dfrac{x-b}{a+b}=\dfrac{x+a}{a-b}.$

7. $\dfrac{m^2x+n}{mx}-\dfrac{n^2x+m}{nx}=\dfrac{m-n}{mnx}.$

8. $\dfrac{x-a}{b}+\dfrac{x-b}{c}+\dfrac{x-c}{a}=\dfrac{bc(x+b)-ab^2-a^2c+abx}{abc}.$

9. $\dfrac{5}{2x+5m}-\dfrac{2}{3x-4m}=\dfrac{3m}{6x^2+7mx-20m^2}.$

10. $\dfrac{a+b}{x}+\dfrac{a-2b}{x+a}=\dfrac{(2a-b)x+3ab}{x^2-a^2}.$

11. $\dfrac{bx}{a}-\dfrac{a^2+b^2}{a^2}=\dfrac{a^2}{b^2}-\dfrac{x(a-b)}{b}.$

12. $\dfrac{a}{x-a}-\dfrac{b}{x-b}=\dfrac{a-b}{x-c}.$

13. $\dfrac{a(x-a)}{x-b}+\dfrac{b(x-b)}{x-a}=a+b.$

14. $(a+b)(x-a+b)-(a-b)x+a^2-b^2=2a(x+a-b).$

15. $(x+p+q)(x-p+q)+q^2=(x-p)(x+q).$

16. $\dfrac{4x+3n}{x+2n}+\dfrac{4x-5n}{3n-x}=\dfrac{10n^2}{x^2-nx-6n^2}.$

17. $\dfrac{3x}{2}-\dfrac{5ax-2b}{4a}=\dfrac{a+3bx}{8b}-\dfrac{ax+2a^2-4b}{16ab}.$

18. $\dfrac{a}{x+b}-\dfrac{b}{x+a}=\dfrac{a-b}{x+a+b}.$

19. $\dfrac{x+a}{x-2a}-\dfrac{x-a}{x+3a}+\dfrac{2ax-19a^2}{x^2+ax-6a^2}=0.$

20. $\dfrac{1}{x-2a}-\dfrac{1}{6x+a}=\dfrac{7}{3x-8a}-\dfrac{3}{2x-3a}.$

21. $\dfrac{4}{x-4n}-\dfrac{1}{x+n}=\dfrac{4}{x+4n}-\dfrac{1}{x+3n}.$

22. $\dfrac{x^2-2ax+a^2}{x^2-2ax-3a^2}+\dfrac{x^2+ax-2a^2}{x^2+ax+2a^2}=2.$

23. $\dfrac{x+a}{x-a}+\dfrac{x+b}{x-b}+\dfrac{x-a-b}{x+a+b}=3.$

24. $x^3+(x-a)^3+(x-b)^3=3x(x-a)(x-b).$

SOLUTION OF EQUATIONS INVOLVING DECIMALS

160. *Ex.* Solve the equation

$$.2x+.001-.03x=.113x-.0161.$$

Transposing, $.2x-.03x-.113x=-.0161-.001.$

Uniting terms, $.057x=-.0171.$

Dividing by .057, $x=-.3.$

EXERCISE 61

Solve the following equations:

1. $7.98x-3.75=.23x+.125.$

2. $3x+.052-7.8x=.04-5.82x-.0696.$

3. $.05v-1.82-.7v=.008v-.504.$

(Here, v represents the unknown number.)

4. $.73x+8.86=.6(2.3x-.4).$

5. $.07(8x-5.7)=.8(5x+.86)+1.321.$

6. $3.2x-.84+\dfrac{.62x-.858}{.9}=.9x.$

7. $\dfrac{6.15x+.67}{x}-\dfrac{.6x-.81}{3x}=\dfrac{5}{4}.$

8. $\dfrac{.9x-2.84}{.3}-\dfrac{.8x-6.52}{.5}=7.4.$

9. $20.1x-\dfrac{.714x-3.189}{.6}=\dfrac{x-2}{.03}-.135.$

PROBLEMS INVOLVING LINEAR EQUATIONS

161. The following problems lead both to integral and fractional equations; the former being somewhat more difficult than those of Exercise 24.

1. A can do a piece of work in 8 days which B can perform in 10 days. In how many days can it be done by both working together?

Let $x =$ the number of days required.

Then, $\frac{1}{x} =$ the part both can do in one day.

Also, $\frac{1}{8} =$ the part A can do in one day,

and $\frac{1}{10} =$ the part B can do in one day.

By the conditions, $\frac{1}{8} + \frac{1}{10} = \frac{1}{x}.$

Clearing of fractions, $5x + 4x = 40$, or $9x = 40$.

Whence, $x = 4\frac{4}{9}$, the number of days required.

2. The second digit of a number exceeds the first by 2; and if the number, increased by 6, be divided by the sum of its digits, the quotient is 5. Find the number.

Let $x =$ the first digit.

Then, $x + 2 =$ the second digit,

and $2x + 2 =$ the sum of the digits.

The number itself is equal to 10 times the first digit, plus the second.

Then, $10x + (x + 2)$, or $11x + 2 =$ the number.

By the conditions, $\frac{11x + 2 + 6}{2x + 2} = 5.$

Whence, $11x + 8 = 10x + 10$, and $x = 2$.

Then, $11x + 2 = 24$, the number required.

3. Divide 44 into two parts such that one divided by the other shall give 2 as a quotient and 5 as a remainder.

Let $n =$ the divisor.

Then, $44 - n =$ the dividend.

Now since the dividend is equal to the product of the divisor and quotient, plus the remainder, we have

$$44 - n = 2n + 5, \text{ whence } -3n = -39.$$

Then, $n = 13$, the divisor,

and $44 - n = 31$, the dividend.

4. Two persons, A and B, 63 miles apart, start at the same time and travel towards each other. A travels at the rate of 4 miles an hour, and B at the rate of 3 miles an hour. How far will each have travelled when they meet?

Let $4x =$ the number of miles that A travels.

Then, $3x =$ the number of miles that B travels.

By the conditions, $4x + 3x = 63.$

Then, $7x = 63$, and $x = 9$.

Whence, $4x = 36$, the number of miles that A travels,

and $3x = 27$, the number of miles that B travels.

It is often advantageous, as in Ex. 4, to represent the unknown number by some *multiple* of x instead of by x itself.

5. At what time between 3 and 4 o'clock are the hands of a watch opposite to each other?

Let $x =$ the number of minute-spaces passed over by the minute-hand from 3 o'clock to the required time.

Then, since the hour-hand is 15 minute-spaces in advance of the minute-hand at 3 o'clock, $x - 15 - 30$, or $x - 45$, will represent the number of minute-spaces passed over by the hour-hand.

But the minute-hand moves 12 times as fast as the hour-hand.

Whence, $x = 12(x - 45)$, or $x = 12x - 540$.

Then, $-11x = -540$, and $x = 49\frac{1}{11}$.

Then the required time is $49\frac{1}{11}$ minutes after 3 o'clock.

EXERCISE 62

1. The denominator of a fraction exceeds twice the numerator by 4. If the numerator be increased by 14, and the denominator decreased by 9, the value of the fraction is $\frac{7}{3}$. Find the fraction.

2. Divide 197 into two parts such that the smaller shall be contained in the greater 5 times, with a remainder 23.

3. A piece of work can be done by A in $2\frac{2}{3}$ hours, and by B in $4\frac{1}{2}$ hours; in how many hours can the work be done by both working together?

4. The second digit of a number of two figures exceeds the first by 5; and if the number, increased by 6, be divided by the sum of the digits, the quotient is 4. Find the number.

5. At what time between 12 and 1 o'clock are the hands of a watch opposite to each other?

6. At what time between 7 and 8 o'clock is the minute-hand of a watch 10 minutes in advance of the hour-hand?

7. A piece of work can be done by A and B working together in 10 days. After working together 7 days, A leaves, and B finishes the work in 9 days. How long will A alone take to do the work?

8. Divide 54 into two parts such that twice the smaller shall be 3 times as much above 29 as 4 times the greater is below 143.

9. At what time between 8 and 9 o'clock are the hands of a watch together?

10. The numerator of a fraction exceeds the denominator by 5. If the numerator be decreased by 9, and the denominator increased by 6, the sum of the resulting fraction and the given fraction is 2. Find the fraction.

11. At what time between 2 and 3 o'clock is the minute-hand of a watch 5 minutes behind the hour-hand?

12. The second digit of a number of two figures is $\frac{1}{3}$ the first; and if the number be divided by the difference of its digits, the quotient is 15, and the remainder 3. Find the number.

13. A garrison of 700 men has provisions for 11 days. After 3 days, a certain number of men leave, and the provisions last 10 days after this time. How many men leave?

14. A woman buys a certain number of eggs for \$1.05; she finds that 7 eggs cost as much more than 18 cents as 8 eggs cost less than 27 cents. How many eggs did she buy?

15. The width of a field is $\frac{2}{3}$ its length. If the width were increased by 5 feet, and the length by 10 feet, the area would be increased by 400 square feet. Find the dimensions.

16. After A has travelled 7 hours at the rate of 10 miles in 3 hours, B sets out to overtake him, travelling at the rate of 9 miles in 2 hours. How far will each have travelled when B overtakes A?

17. The first digit of a number of three figures is $\frac{3}{4}$ the second, and exceeds the third digit by 2. If the number be divided by the sum of its digits, the quotient is 38. Find the number.

18. A, B, and C divide coins in the following way: as often as A takes 5, B takes 4, and as often as A takes 6, C takes 7. After the coins have been divided, A has 29 fewer than B and C together. How many coins were there?

19. A can do a piece of work in $3\frac{1}{3}$ hours, B in $3\frac{3}{4}$ hours, and C in $3\frac{3}{5}$ hours. In how many hours can it be done by all working together?

20. A man walks $13\frac{1}{2}$ miles, and returns in an hour less time by a carriage, whose rate is $\frac{3}{2}$ as great as his rate of walking. Find his rate of walking.

21. At what times between 4 and 5 o'clock are the hands of a watch at right angles to each other?

22. A man borrows a certain sum, paying interest at the rate of 5%. After repaying \$180, his interest rate on the balance is reduced to $4\frac{1}{4}\%$, and his annual interest is now less by \$10.80. Find the sum borrowed.

23. The digits of a certain number are three consecutive numbers, of which the middle digit is the greatest, and the first digit the least. If the number be divided by the sum of its digits, the quotient is $\frac{229}{7}$. Find the number.

24. A certain number of apples were divided between three boys. The first received one-half the entire number, with one apple additional, the second received one-third the remainder, with one apple additional, and the third received the remainder, 7. How many apples were there?

25. A freight train runs 6 miles an hour less than a passenger train. It runs 80 miles in the same time that the passenger train runs 112 miles. Find the rate of each train.

26. A and B each fire 40 times at a target; A's hits are one-half as numerous as B's misses, and A's misses exceed by 15 the number of B's hits. How many times does each hit the target?

27. A freight train travels from A to B at the rate of 12 miles an hour. After it has been gone $3\frac{1}{2}$ hours, an express train leaves A for B, travelling at the rate of 45 miles an hour, and reaches B 1 hour and 5 minutes ahead of the freight. Find the distance from A to B, and the time taken by the express train.

28. A tank has three taps. By the first it can be filled in 3 hours 10 minutes, by the second it can be filled in 4 hours 45 minutes, and by the third it can be emptied in 3 hours 48 minutes. How many hours will it take to fill it if all the taps are open?

29. A man invested a certain sum at $3\frac{3}{4}\%$, and $\frac{11}{15}$ this sum at $4\frac{1}{4}\%$; after paying an income tax of 5%, his net annual income is \$195.70. How much did he invest in each way?

30. A train leaves A for B, 210 miles distant, travelling at the rate of 28 miles an hour. After it has been gone 1 hour and 15 minutes, another train starts from B for A, travelling at the rate of 22 miles an hour. How many miles from B will they meet?

31. A can do a piece of work in $\frac{2}{3}$ as many days as B, and B can do it in $\frac{4}{5}$ as many days as C. Together they can do the work in $3\frac{7}{11}$ days. In how many days can each alone do the work?

32. A vessel runs at the rate of $11\frac{2}{3}$ miles an hour. It takes just as long to run 23 miles up stream as 47 miles down stream. Find the rate of the stream.

33. A man starts from his home to catch a train at the rate of one yard in a second, and arrives 2 minutes late. If he had walked at the rate of 4 yards in 3 seconds, he would have been $3\frac{1}{2}$ minutes too early. Find the distance to the station.

34. A crew has bread for a voyage of 50 days, at $1\frac{1}{2}$ lb. each a day. After 20 days, 7 men are lost in a storm, and the remainder of the crew have a daily allowance of $1\frac{4}{5}$ lb. for the balance of the voyage. Find the original number of the crew.

35. A man invests \$230 at $4\frac{1}{2}\%$. He then invests a certain part of a like sum at $3\frac{1}{3}\%$, and the balance at $5\frac{1}{4}\%$, and obtains the same income. How much does he invest at each rate?

36. At what times between 5 and 6 o'clock do the hands of a watch make an angle of 45°?

37. At a certain time between 12 noon and 12.30 P.M., the distance between the hands is $\frac{2}{3}$ as great as it is 10 minutes later. Find the time.

38. A woman sells half an egg more than half her eggs. She then sells half an egg more than half her remaining eggs. A third time she does the same, and now has 3 eggs left. How many had she at first?

39. A merchant increases his capital annually by $\frac{1}{3}$ of itself. He adds to his capital \$300 at the end of the first year, and \$350 at the end of the second; and finds at the end of the third year that his capital is $\frac{68}{27}$ of his original capital. Find his original capital.

40. A and B together can do a piece of work in $5\frac{1}{7}$ days, B and C together in $6\frac{2}{3}$ days, and C and A together in $5\frac{5}{8}$ days. In how many days can it be done by each working alone?

41. A fox is pursued by a hound, and has a start of 77 of her own leaps. The fox makes 5 leaps while the hound makes 4; but the hound in 5 leaps goes as far as the fox in 9. How many leaps does each make before the hound catches the fox?

42. A man puts a certain sum into a savings bank paying 4 % interest. At the end of a year he deposits the interest, receiving interest on the entire amount. At the end of a second year and a third year he does the same, and now has \$2812.16 in the bank. What was his original deposit?

PROBLEMS IN PHYSICS

1. The density of a substance is defined as the number of grams in one cubic centimeter. Hence the total number of grams, M, in any body is equal to its density, D, multiplied by its volume, V; or, to state this relation algebraically,

$$M = DV,$$

V being given in cubic centimeters, and D in grams.

Two blocks, one of iron and one of copper, weigh the same number of grams; the iron has a volume of 10 cubic centimeters and a density of 7.4; the copper has a density of 8.9. Find the volume of the copper block.*

2. When 100 grams of alcohol, of density .8, is poured into a cylindrical vessel, it is found to fill it to a depth of 10 centimeters. Find the area of the base of the cylinder in square centimeters.

3. A cylindrical iron bar, 2 centimeters in diameter, has a mass of 3 kilograms. Find the length of the bar.

Let $\pi = 3\frac{1}{7}$.

* Metric tables are found on page 458.

4. When a body is weighed under water, it is found to be buoyed up by a force equal to the weight of the water which it displaces.

If a boy can exert a lifting force of 120 pounds, how heavy a stone can he lift to the surface of a pond, if the density of stone is 2.5 and that of water 1?

5. When a straight bar is supported at some point, o (Fig. 1), and masses m_1, m_2, etc., are hung from the bar as indicated in the figure, it is found that when the bar is in equilibrium, the following relation always holds,

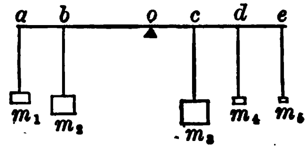


FIG. 1.

$$m_1 \times ao + m_2 \times bo = m_3 \times co + m_4 \times do + m_5 \times eo.$$

If a teeter board is 10 feet long, where must the support be placed in order that a 70-pound boy at one end may balance a 60-pound boy on the other end plus a 40-pound boy 3 feet from the other end?

6. A bar 40 inches long is in equilibrium when weights of 6 pounds and 9 pounds hang from its two ends. Find the position of the support.

7. If in Fig. 1, $ao = 100$, $bo = 40$, $co = 30$, $do = 60$, $eo = 110$, and if $m_1 = 40$, $m_2 = 60$, $m_3 = 60$, $m_4 = 15$, and $m_5 = 5$, where must a mass of 100 be placed in order to produce equilibrium?

8. A gas expands $\frac{1}{273}$ of its volume at 0° centigrade for each degree of rise in its temperature; *i.e.*, the volume, V_t, at any temperature, t, is connected with the volume, V_0, at the temperature 0° centigrade by the equation

$$V_t = V_0 + \tfrac{1}{273} V_0 t,$$

or

$$V_t = V_0(1 + \tfrac{1}{273} t).$$

To what volume will 100 cubic centimeters of air at 0° expand when the temperature rises to 50° centigrade?

9. To what volume will 100 cubic centimeters of air at 50° centigrade contract when the temperature falls to 0° centigrade?

10. To what volume will 100 cubic centimeters of air at 50° expand when the temperature changes to 75°?

11. When a body in motion collides with a body at rest, the momentum of the first body (*i.e.*, the product of its mass, m_1, by its original velocity, v_1) is found to be in every case exactly equal to the total momentum of the two bodies after collision (*i.e.*, to the product of the mass, m_2, of the second body times the velocity, v_2, which it acquires, plus the product of m_1 by the velocity, v_3, which it retains after the collision). The algebraic statement of this relation is

$$m_1v_1 = m_2v_2 + m_1v_3.$$

A billiard ball, the mass of which is 50 grams, and which was moving at a velocity of 1500 centimeters a second, collided with another ball at rest which weighed 30 grams. In the collision the first ball imparted to the second a velocity of 1600 centimeters per second. Find the velocity of the first ball after the collision.

PROBLEMS INVOLVING LITERAL EQUATIONS

162. *Prob.* Divide a into two parts such that m times the first shall exceed n times the second by b.

Let $\qquad x = \text{one part.}$

Then, $\qquad a - x = \text{the other part.}$

By the conditions, $\qquad mx = n(a - x) + b.$

$$mx = an - nx + b.$$

$$mx + nx = an + b.$$

$$x(m + n) = an + b.$$

Whence, $$x = \frac{an + b}{m + n}, \text{ the first part.} \qquad (1)$$

And, $$a - x = a - \frac{an + b}{m + n} = \frac{am + an - an - b}{m + n}$$

$$= \frac{am - b}{m + n}, \text{ the other part.} \qquad (2)$$

The results can be used as *formulæ* for solving any problem of the above form.

Thus, let it be required to divide 25 into two parts such that 4 times the first shall exceed 3 times the second by 37.

Here, $a = 25$, $m = 4$, $n = 3$, and $b = 37$.

Substituting these values in (1) and (2),

the first part $= \frac{25 \times 3 + 37}{7} = \frac{75 + 37}{7} = \frac{112}{7} = 16,$

and the second part $= \frac{25 \times 4 - 37}{7} = \frac{100 - 37}{7} = \frac{63}{7} = 9.$

EXERCISE 63

1. Divide a into two parts whose quotient shall be m.

2. If A can do a piece of work in m hours, and A and B together in n hours, in how many hours can B alone do the work?

3. Divide a into two parts such that the sum of one-mth the first and one-nth the second shall equal b.

4. A courier who travels a miles a day is followed by another who travels b miles a day. How many days must the second start after the first to overtake him after c days?

5. Divide a into three parts such that the first shall be one-mth the second and one-nth the third.

6. The length of a field is m times its width. If the length were increased by a feet, and the width by b feet, the area would be increased by c square feet. Find the dimensions of the field.

7. A courier who travels a miles a day is followed after b days by another. How many miles a day must the second courier travel to overtake the first after c days?

8. If A can do a piece of work in a hours, B in b hours, C in c hours, and D in d hours, how many hours will it take to do the work if all work together?

9. A vessel can be filled by two taps in a and b minutes, respectively, and emptied by a third in c minutes. How many minutes will it take to fill the tank if all the taps are open?

10. Divide a into two parts such that one shall be m times as much above b as the other lacks of c.

11. A can do a piece of work in one-mth as many days as B, and B can do it in one-nth as many as C. If they can do the work in p days, working together, in how many days can each alone do the work?

12. A was m times as old as B a years ago, and will be n times as old as B in b years. Find their ages at present.

13. How many minutes after n hours after 12 o'clock will the hands of a watch be together?

14. A and B together can do a piece of work in a hours, B and C together in b hours, and A, B, and C together in c hours. In how many hours can each alone do the work?

15. How many minutes after 2 o'clock will the minute-hand of a watch be n minutes in advance of the hour-hand?

16. A and B together can do a piece of work in m days, B and C together in n days, and C and A together in p days. How many days will it take to do the work if all work together?

17. A sum of money, amounting to m dollars, consists entirely of quarters and dimes, there being n more dimes than quarters. How many are there of each?

XII. SIMULTANEOUS LINEAR EQUATIONS

CONTAINING TWO OR MORE UNKNOWN NUMBERS

163. An equation containing two or more unknown numbers is satisfied by an indefinitely great number of sets of values of these numbers.

Consider, for example, the equation $x+y=5$.

Putting $x=1$, we have $1+y=5$, or $y=4$.

Putting $x=2$, we have $2+y=5$, or $y=3$; etc.

Thus the equation is satisfied by the sets of values

$$x=1,\ y=4,$$

and $$x=2,\ y=3;\ \text{etc.}$$

An equation which is satisfied by an indefinitely great number of sets of values of the unknown numbers involved, is called an **Indeterminate Equation.**

164. Consider the equations

$$\begin{cases} x+y=5, & (1)\\ 2x+2y=10. & (2)\end{cases}$$

Equation (1) can be made to take the form of (2) by multiplying both members by 2; then, every set of values of x and y which satisfies one of the equations also satisfies the other.

Such equations are called *equivalent.*

Again, consider the equations

$$\begin{cases} x+y=5, & (3)\\ x-y=3. & (4)\end{cases}$$

In this case, it is not true that every set of values of x and y which satisfies one of the equations also satisfies the other; thus, equation (3) is satisfied by the set of values $x=3$, $y=2$, which does not satisfy (4).

If two equations, containing two or more unknown numbers, are not equivalent, they are called **Independent.**

165. Consider the equations

$$\begin{cases} x+y=5, & (1) \\ x+y=6. & (2) \end{cases}$$

It is evidently impossible to find a set of values of x and y which shall satisfy both (1) and (2).

Such equations are called **Inconsistent.**

166. A system of equations is called **Simultaneous** when each contains two or more unknown numbers, and every equation of the system is satisfied by the same set, or sets, of values of the unknown numbers; thus, each equation of the system

$$\begin{cases} x+y=5, \\ x-y=3, \end{cases}$$

is satisfied by the set of values $x=4$, $y=1$.

A **Solution** of a system of simultaneous equations is a set of values of the unknown numbers which satisfies every equation of the system; to *solve* a system of simultaneous equations is to find its solutions.

167. Two independent simultaneous equations of the form $ax+by=c$ may be solved by combining them in such a way as to form a single equation containing but *one* unknown number.

This operation is called **Elimination.**

ELIMINATION BY ADDITION OR SUBTRACTION

168. 1. Solve the equations $\begin{cases} 5x-3y=19. & (1) \\ 7x+4y=\ \ 2. & (2) \end{cases}$

Multiplying (1) by 4,	$20x-12y=76.$	(3)
Multiplying (2) by 3,	$21x+12y=\ \ 6.$	(4)
Adding (3) and (4),	$41x=82.$	(5)
Whence,	$x=\ \ 2.$	(6)
Substituting $x=2$ in (1),	$10-3y=19.$	(7)
Whence,	$-3y=9$, or $y=-3.$	(8)

The above is an example of elimination by *addition.*

We speak of *adding* a system of equations when we mean placing the sum of the first members equal to the sum of the second members.

Abbreviations of this kind are frequent in Algebra; thus we speak of *multiplying* an equation when we mean multiplying each of its terms.

2. Solve the equations $\begin{cases} 15x+8y=1. & (1) \\ 10x-7y=-24. & (2) \end{cases}$

Multiplying (1) by 2, $30x+16y=2.$ (3)

Multiplying (2) by 3, $30x-21y=-72.$ (4)

Subtracting (4) from (3), $37y=74$, and $y=2$.

Substituting $y=2$ in (1), $15x+16=1$.

Whence, $15x=-15$, and $x=-1$.

The above is an example of elimination by *subtraction*.

From the above examples, we have the following rule:

If necessary, multiply the given equations by such numbers as will make the coefficients of one of the unknown numbers in the resulting equations of equal absolute value.

Add or subtract the resulting equations according as the coefficients of equal absolute value are of unlike or like sign.

If the coefficients which are to be made of equal absolute value are prime to each other, each may be used as the multiplier for the other equation; but if they are not prime to each other, such multipliers should be used as will produce their lowest common multiple.

Thus, in Ex. 1, to make the coefficients of y of equal absolute value, we multiply (1) by 4 and (2) by 3; but in Ex. 2, to make the coefficients of x of equal absolute value, since the L.C.M. of 10 and 15 is 30, we multiply (1) by 2 and (2) by 3.

EXERCISE 64

In several examples in the following set other letters than x and y are used to represent unknown numbers.

Solve by the method of addition or subtraction:

1. $\begin{cases} 6x+5y=28. \\ 4x+y=14. \end{cases}$

2. $\begin{cases} x-5y=-21. \\ 3x-8y=-35. \end{cases}$

3. $\begin{cases} 2x-3y=19. \\ 7x+4y=23. \end{cases}$

4. $\begin{cases} 11x-15y=-7. \\ 5y+9x=-23. \end{cases}$

5. $\begin{cases} 7x - 3y = 10. \\ 3x - 5y = -5. \end{cases}$

6. $\begin{cases} 15x + 8y = 3. \\ 6x - 12y = 5. \end{cases}$

7. $\begin{cases} 10x + 15y = -22. \\ 7x + 20y = -4. \end{cases}$

8. $\begin{cases} 4x - 8y = -3. \\ 11x + 5y = -15. \end{cases}$

9. $\begin{cases} 9x - 14y = -30. \\ 21x + 13y = 67. \end{cases}$

10. $\begin{cases} 13m - 7n = 15. \\ 8m - 4n = 9. \end{cases}$

11. $\begin{cases} 6x + 11y = 31. \\ 6y - 11x = 74. \end{cases}$

12. $\begin{cases} 9u + 6v = -16. \\ 13u + 7v = -22. \end{cases}$

13. $\begin{cases} 12x - 11y = 19. \\ 12y - 11x = -27. \end{cases}$

14. $\begin{cases} 24p - 7t = 52. \\ 18p + 13t = -34. \end{cases}$

15. $\begin{cases} 19x + 20y = -35. \\ 21x + 16y = -57. \end{cases}$

16. $\begin{cases} 12x + 11y = 172. \\ 28x - 17y = 60. \end{cases}$

ELIMINATION BY SUBSTITUTION

169. *Ex.* Solve the equations $\begin{cases} 7x - 9y = 15. & (1) \\ 8y - 5x = -17. & (2) \end{cases}$

Transposing $-5x$ in (2), $\quad 8y = 5x - 17.$

Whence, $\quad y = \dfrac{5x - 17}{8}.$ (3)

Substituting in (1), $\quad 7x - 9\left(\dfrac{5x - 17}{8}\right) = 15.$ (4)

Clearing of fractions, $\quad 56x - 9(5x - 17) = 120.$

Or, $\quad 56x - 45x + 153 = 120.$

Uniting terms, $\quad 11x = -33.$

Whence, $\quad x = -3.$ (5)

Substituting $x = -3$ in (3), $\quad y = \dfrac{-15 - 17}{8} = -4.$ (6)

From the above example, we have the following rule:

From one of the given equations find the value of one of the unknown numbers in terms of the other, and substitute this value in place of that number in the other equation.

EXERCISE 65

Solve by the method of substitution:

1. $\begin{cases} x+2y=11. \\ 3x+5y=29. \end{cases}$

2. $\begin{cases} 2x+y=8. \\ 7x-4y=43. \end{cases}$

3. $\begin{cases} 5x-6y=-9. \\ 3x-5y=-4. \end{cases}$

4. $\begin{cases} 3x+7y=-12. \\ 9y-6x=1. \end{cases}$

5. $\begin{cases} 5p+2r=-4. \\ 6p-11r=-45. \end{cases}$

6. $\begin{cases} 3x-5y=38. \\ 3y-5x=-26. \end{cases}$

7. $\begin{cases} 25x-12y=-19. \\ 10x+4y=-1. \end{cases}$

8. $\begin{cases} 5x+6y=-5. \\ 10x+9y=-6. \end{cases}$

9. $\begin{cases} 8e-3f=47. \\ 6e-7f=21. \end{cases}$

10. $\begin{cases} 4x-11y=-71. \\ 9x+8y=4. \end{cases}$

11. $\begin{cases} 6x+12y=41. \\ 3y-4x=-9. \end{cases}$

12. $\begin{cases} 7x+6y=-13. \\ 9x+10y=-11. \end{cases}$

13. $\begin{cases} 8m-15v=18. \\ 12m+6v=-11. \end{cases}$

14. $\begin{cases} 9x+8y=57. \\ 6x+7y=48. \end{cases}$

15. $\begin{cases} 18x-10y=29. \\ 15y-14x=-24. \end{cases}$

16. $\begin{cases} 7x-9y=-22. \\ 11x+4y=-89. \end{cases}$

ELIMINATION BY COMPARISON

170. *Ex.* Solve the equations $\begin{cases} 2x-5y=-16. & (1) \\ 3x+7y=5. & (2) \end{cases}$

Transposing $-5y$ in (1), $2x=5y-16.$

Whence, $x=\dfrac{5y-16}{2}.$ (3)

Transposing $7y$ in (2), $3x=5-7y.$

Whence, $x=\dfrac{5-7y}{3}.$ (4)

Equating values of x, $\dfrac{5y-16}{2}=\dfrac{5-7y}{3}.$ (5)

Clearing of fractions, $15y - 48 = 10 - 14y.$

Transposing, $29y = 58.$

Whence, $y = 2.$ (6)

Substituting $y = 2$ in (3), $x = \frac{10 - 16}{2} = -3.$ (7)

From the above example, we have the following rule:

From each of the given equations, find the value of the same unknown number in terms of the other, and place these values equal to each other.

EXERCISE 66

Solve by the method of comparison:

1. $\begin{cases} 2x + 3y = 14. \\ x + 4y = 17. \end{cases}$

2. $\begin{cases} 6x - y = 27. \\ 8y - 3x = -36. \end{cases}$

3. $\begin{cases} 3x + 2y = -31. \\ 7x - 3y = -34. \end{cases}$

4. $\begin{cases} 5x - 8y = -46. \\ 2x + 3y = -6. \end{cases}$

5. $\begin{cases} 5x - 2y = 4. \\ 4x + 7y = 29. \end{cases}$

6. $\begin{cases} 7r - 3s = -18. \\ 4r - 5s = -7. \end{cases}$

7. $\begin{cases} 3x - 8y = -13. \\ 6x - 4y = -5. \end{cases}$

8. $\begin{cases} 8x - 7y = -6. \\ 6x + 11y = -37. \end{cases}$

9. $\begin{cases} 9x + 5y = -1. \\ 9y - 6x = 13. \end{cases}$

10. $\begin{cases} 8x + 5y = 5. \\ 12x + 10y = 7. \end{cases}$

11. $\begin{cases} 10h + 6k = -15. \\ 14k - 15h = -58. \end{cases}$

12. $\begin{cases} 8x + 9y = 8. \\ 10x + 12y = 11. \end{cases}$

13. $\begin{cases} 11d + 16t = 64. \\ 7d - 12t = 13. \end{cases}$

14. $\begin{cases} 8x - 7y = -68. \\ 16y - 5x = 89. \end{cases}$

15. $\begin{cases} 9x - 6y = -17. \\ 7x + 15y = -46. \end{cases}$

16. $\begin{cases} 8x + 5y = 49. \\ 13x + 6y = 86. \end{cases}$

171. If the given equations are not in the form $ax + by = c$, they should first be reduced to this form, when they may be solved by either method of elimination.

1. Solve the equations $\begin{cases} \dfrac{7}{x+3} - \dfrac{3}{y+4} = 0. & (1) \\ x(y-2) - y(x-5) = -13. & (2) \end{cases}$

Multiplying each term of (1) by $(x+3)(y+4)$,

$$7y + 28 - 3x - 9 = 0, \text{ or } 7y - 3x = -19. \qquad (3)$$

From (2), $xy - 2x - xy + 5y = -13$, or $5y - 2x = -13$. (4)

Multiplying (3) by 2, $14y - 6x = -38$. (5)

Multiplying (4) by 3, $15y - 6x = -39$. (6)

Subtracting (5) from (6), $y = -1$.

Substituting in (4), $-5 - 2x = -13$.

Whence, $-2x = -8$, or $x = 4$.

In solving fractional simultaneous equations, we reject any solution which does not satisfy the given equations.

2. Solve the equations $\begin{cases} 2x + 3y = 13. & (1) \\ \dfrac{1}{x-2} + \dfrac{1}{y-3} = 0. & (2) \end{cases}$

Multiplying each term of (2) by $(x-2)(y-3)$, we have

$$y - 3 + x - 2 = 0, \text{ or } y = -x + 5. \qquad (3)$$

Substituting in (1), $2x - 3x + 15 = 13$, or $x = 2$.

Substituting in (3), $y = -2 + 5 = 3$.

This solution satisfies the first given equation, but not the second; then it must be rejected.

EXERCISE 67

Solve the following:

1. $\begin{cases} \dfrac{2x}{5} - \dfrac{5y}{6} = -\dfrac{1}{2}. \\ \dfrac{x}{6} + \dfrac{5y}{9} = \dfrac{5}{2}. \end{cases}$

2. $\begin{cases} \dfrac{11x - 3y}{11} = \dfrac{3x + y}{8}. \\ 8x - 5y = 1. \end{cases}$

3. $\begin{cases} \dfrac{4x - 3y}{14} - \dfrac{x - 6y}{9} = -1. \\ 2x + 3y = -10. \end{cases}$

4. $\begin{cases} \dfrac{2e + t + 6}{e - 2t - 3} = -\dfrac{2}{7}. \\ 5e + 2t = -7. \end{cases}$

5. $\begin{cases} 3x-4y=-11. \\ \dfrac{2}{x+5}-\dfrac{5}{y+1}=0. \end{cases}$

6. $\begin{cases} x-\dfrac{4y-9}{11}=5. \\ \dfrac{9}{2}-\dfrac{x+5}{3}=-3y. \end{cases}$

7. $\begin{cases} (2x-1)(y-4)-(x-5)(2y+5)=121. \\ 4x-3y=-29. \end{cases}$

8. $\begin{cases} \dfrac{7}{x-3}-\dfrac{8}{y-5}=0. \\ \dfrac{9}{2x-1}-\dfrac{5}{3y+4}=0. \end{cases}$

9. $\begin{cases} \dfrac{x+11}{7}+\dfrac{y-6}{5}=-4. \\ \dfrac{x-1}{2}-\dfrac{y+4}{10}=-45. \end{cases}$

10. $\begin{cases} \dfrac{d-2n}{3d+n+3}=-\dfrac{1}{5}. \\ \dfrac{d+3n}{d+4n-7}=\dfrac{7}{11}. \end{cases}$

11. $\begin{cases} .08x+.9y=.048. \\ .3x-.35y=.478. \end{cases}$

12. $\begin{cases} \dfrac{2x-3y}{4}+\dfrac{4x+6y}{3}=-\dfrac{1}{2}. \\ \dfrac{5x+2y}{2}+\dfrac{7y-3x}{5}=\dfrac{39}{10}. \end{cases}$

13. $\begin{cases} \dfrac{x+y}{x-y}=-\dfrac{1}{10}. \\ \dfrac{3x+8}{y-4}=\dfrac{6x-1}{2y+3}. \end{cases}$

14. $\begin{cases} 5x-\frac{1}{3}(3x-2y+5)=11. \\ \frac{5}{6}(x-4y)-\frac{4}{9}(x-y)=16. \end{cases}$

15. $\begin{cases} \dfrac{8x-3}{4}+\dfrac{y-5}{3}=\dfrac{y}{6}. \\ x-\dfrac{y-7}{5}=\dfrac{5}{8}. \end{cases}$

16. $\begin{cases} \frac{1}{2}(p-q)-\frac{1}{5}(p-3q)=q-3. \\ \frac{3}{4}(p-q)+\frac{5}{6}(p+q)=18. \end{cases}$

17. $\begin{cases} \dfrac{6x+5y}{16}-\dfrac{3x-4y}{5}=x+y+3. \\ \dfrac{5x-4y}{5x+4y}=-\dfrac{3}{13}. \end{cases}$

18. $\begin{cases} \dfrac{2x+5y+1}{5}-\dfrac{3x+y-3}{8}=-x+2y-2. \\ \dfrac{x-4y+6}{8x-2y-18}=-\dfrac{1}{4}. \end{cases}$

19. $\begin{cases} 3x-y-1-\frac{1}{6}\left[3x+5\left(2y-\frac{3}{4}\right)\right]=0. \\ \frac{1}{3}(3x+y)-\frac{1}{4}\left(x-\frac{y}{3}\right)=\frac{5}{12}. \end{cases}$

20. $\begin{cases} \frac{5x-7y+2}{3}-\frac{3x-4y+7}{4}=y+4. \\ \frac{7x-3y+4}{4}-\frac{6x-5y+7}{5}=x-2. \end{cases}$

21. $\begin{cases} \dfrac{\frac{5x}{12}-\frac{2y}{3}}{\frac{7}{4}}-\dfrac{\frac{x}{3}-\frac{3y}{2}}{\frac{23}{2}}=2. \\ x-2y+4(2x-y)=0. \end{cases}$

22. $\begin{cases} \frac{14g+1}{3}-\frac{10v-3}{5}=2. \\ \frac{21g+1}{2}-\frac{5v+2}{3}=\frac{63g-130v}{21}. \end{cases}$

23. $\begin{cases} \frac{3x+1}{7}-\frac{2x+y}{2}+\frac{x+2y}{8}=0. \\ \frac{4x-2}{3}+\frac{5x+4y}{2}=\frac{x-y}{5}. \end{cases}$

24. $\begin{cases} \frac{2x+3}{4y+5}-\frac{3x+4}{6y+7}-\frac{17}{2(4y+5)(6y+7)}=0. \\ (x-1)(y+2)-(x+3)(y+4)=12. \end{cases}$

25. $\begin{cases} \frac{.08x+.35}{.15}-\frac{.13y+.29}{.6}=.32x+.17y+.21. \\ \frac{.02x+.17}{.9}-\frac{.08y-.47}{.3}=0. \end{cases}$

172. Solution of Literal Simultaneous Equations.

In solving *literal* simultaneous linear equations, the method of elimination by addition or subtraction is usually to be preferred.

Ex. Solve the equations $\begin{cases} ax+by=c. & (1)\\ a'x+b'y=c'. & (2)\end{cases}$

Multiplying (1) by b', $ab'x + bb'y = b'c.$

Multiplying (2) by b, $a'bx + bb'y = bc'.$

Subtracting, $(ab' - a'b)x = b'c - bc'.$

Whence, $$x = \frac{b'c - bc'}{ab' - a'b}.$$

Multiplying (1) by a', $aa'x + a'by = ca'.$ (3)

Multiplying (2) by a, $aa'x + ab'y = c'a.$ (4)

Subtracting (3) from (4), $(ab' - a'b)y = c'a - ca'.$

Whence, $$y = \frac{c'a - ca'}{ab' - a'b}.$$

In solving *fractional* literal simultaneous equations, any solution which does not satisfy the given equations must be rejected. (Compare Ex. 2, § 171.)

EXERCISE 68

Solve the following:

1. $\begin{cases} 5x-6y=8a.\\ 4x+9y=7a.\end{cases}$

2. $\begin{cases} ax+by=1.\\ cx+dy=1.\end{cases}$

3. $\begin{cases} a_1x+a_2y=b_1.\\ a_2x-a_1y=b_2.\end{cases}$

4. $\begin{cases} mx-ny=mn.\\ m'x+n'y=m'n'.\end{cases}$

5. $\begin{cases} \dfrac{2ax-by}{a}=b.\\ \dfrac{x+by}{3a+2}=b.\end{cases}$

6. $\begin{cases} \dfrac{x}{m_1}+\dfrac{y}{m_2}=\dfrac{1}{m_3}.\\ \dfrac{x}{n_1}+\dfrac{y}{n_2}=\dfrac{1}{n_3}.\end{cases}$

7. $\begin{cases} bx-ay=b^2.\\ (a-b)x+by=a^2.\end{cases}$

8. $\begin{cases} \dfrac{m}{n+y}=\dfrac{n}{m-x}.\\ \dfrac{m}{n+x}=\dfrac{n}{m-y}.\end{cases}$

9. $\begin{cases} ax+by=2a.\\ a^2x-b^2y=a^2+b^2.\end{cases}$

10. $\begin{cases} (a+1)x+(a-2)y=3a.\\ (a+3)x+(a-4)y=7a.\end{cases}$

11. $\begin{cases} ab(a-b)x+ab(a+b)y=a^2+2ab-b^2.\\ ax+by=2.\end{cases}$

12. $\begin{cases} m(x+y)+n(x-y)=2. \\ m^2(x+y)-n^2(x-y)=m-n. \end{cases}$

13. $\begin{cases} (a+b)x+(a-b)y=2(a^2+b^2). \\ \dfrac{b}{x-a-b}=\dfrac{a}{y-a+b}. \end{cases}$

14. $\begin{cases} (a+b)x+(a-b)y=2a^2-2b^2. \\ \dfrac{y}{a-b}-\dfrac{x}{a+b}=\dfrac{4ab}{a^2-b^2}. \end{cases}$

15. $\begin{cases} bx+ay=2. \\ ab(a+b)x-ab(a-b)y=a^2+b^2. \end{cases}$

16. $\begin{cases} \dfrac{y-a+b}{x+a+b}=\dfrac{y-a}{x+b}. \\ \dfrac{a-x}{y-b}=\dfrac{b}{a}. \end{cases}$ 17. $\begin{cases} ay-bx=a^2+b^2. \\ (a+b)x+(a-b)y=2a^2-2b^2. \end{cases}$

18. $\begin{cases} (a+b)x+(a-b)y=2a. \\ (a^2-b^2)x+(a^2-b^2)y=2a^2+2b^2. \end{cases}$

173. Certain equations in which the unknown numbers occur in the denominators of fractions may be readily solved without previously clearing of fractions.

Ex. Solve the equations $\begin{cases} \dfrac{10}{x}-\dfrac{9}{y}=8. & (1) \\ \dfrac{8}{x}+\dfrac{15}{y}=-1. & (2) \end{cases}$

Multiplying (1) by 5, $\dfrac{50}{x}-\dfrac{45}{y}=40.$

Multiplying (2) by 3, $\dfrac{24}{x}+\dfrac{45}{y}=-3.$

Adding, $\dfrac{74}{x}=37$, $74=37x$, and $x=2$.

Substituting in (1), $5-\dfrac{9}{y}=8$, $-\dfrac{9}{y}=3$, and $y=-3$.

EXERCISE 69

Solve the following:

1. $\begin{cases} \dfrac{6}{x}+\dfrac{12}{y}=-1. \\ \dfrac{8}{x}-\dfrac{9}{y}=7. \end{cases}$

2. $\begin{cases} \dfrac{9}{x}-3y=4. \\ \dfrac{3}{x}+2y=\dfrac{10}{3}. \end{cases}$

3. $\begin{cases} \dfrac{a}{x}+\dfrac{b}{y}=c. \\ \dfrac{a'}{x}+\dfrac{b'}{y}=c'. \end{cases}$

4. $\begin{cases} \dfrac{9}{d}+\dfrac{14}{s}=-\dfrac{11}{2}. \\ \dfrac{6}{d}+\dfrac{21}{s}=-7. \end{cases}$

5. $\begin{cases} \dfrac{2}{3x}-\dfrac{3}{4y}=\dfrac{1}{12}. \\ \dfrac{5}{4x}-\dfrac{4}{3y}=\dfrac{13}{72}. \end{cases}$

6. $\begin{cases} \dfrac{8}{x}-\dfrac{3}{5y}=-\dfrac{89}{30}. \\ \dfrac{5}{6x}-\dfrac{6}{y}=-\dfrac{59}{18}. \end{cases}$

7. $\begin{cases} \dfrac{a}{bx}+\dfrac{b}{ay}=\dfrac{a+b}{ab}. \\ \dfrac{b}{ax}-\dfrac{a}{by}=\dfrac{b^3-a^3}{a^2b^2}. \end{cases}$

8. $\begin{cases} \dfrac{a}{x-a}+\dfrac{b}{y+b}=1. \\ \dfrac{b}{x-a}+\dfrac{a}{y+b}=1. \end{cases}$

9. $\begin{cases} \dfrac{p-q}{x}-\dfrac{q}{y}=\dfrac{p^2-2pq-q^2}{p(p+q)}. \\ \dfrac{p}{x}-\dfrac{p+q}{y}=\dfrac{-2pq-q^2}{p(p+q)}. \end{cases}$

10. $\begin{cases} \dfrac{3}{2x+y}-\dfrac{24}{x-4y}=-2. \\ \dfrac{7}{2x+y}-\dfrac{16}{x-4y}=-3. \end{cases}$

SIMULTANEOUS LINEAR EQUATIONS CONTAINING MORE THAN TWO UNKNOWN NUMBERS

174. If we have *three* independent simultaneous equations, containing *three* unknown numbers, we may combine any two of them by one of the methods of elimination explained in §§ 169 to 171, so as to obtain a single equation containing only two unknown numbers.

We may then combine the remaining equation with either of the other two, and obtain another equation containing the *same* two unknown numbers.

By solving the two equations containing two unknown numbers, we may obtain their values; and substituting them in either of the given equations, the value of the remaining unknown number may be found.

We proceed in a similar manner when the number of equations and of unknown numbers is greater than three.

The method of elimination by addition or subtraction is usually the most convenient.

In solving *fractional* simultaneous equations, any solution which does not satisfy the given equations must be rejected. (Compare Ex. 2, § 171.)

1. Solve the equations
$$\begin{cases} 6x-4y-7z=17. & (1)\\ 9x-7y-16z=29. & (2)\\ 10x-5y-3z=23. & (3)\end{cases}$$

Multiplying (1) by 3, $18x-12y-21z=51.$

Multiplying (2) by 2, $18x-14y-32z=58.$

Subtracting, $2y+11z=-7.$ (4)

Multiplying (1) by 5, $30x-20y-35z=85.$ (5)

Multiplying (3) by 3, $30x-15y-9z=69.$ (6)

Subtracting (5) from (6) $5y+26z=-16.$ (7)

Multiplying (4) by 5, $10y+55z=-35.$

Multiplying (7) by 2, $10y+52z=-32.$

Subtracting, $3z=-3$, or $z=-1.$

Substituting in (7), $2y-11=-7$, or $y=2.$

Substituting in (1), $6x-8+7=17$, or $x=3.$

In certain cases the solution may be abridged by aid of the artifice which is employed in the following example:

2. Solve the equations
$$\begin{cases} u+x+y=6. & (1)\\ x+y+z=7. & (2)\\ y+z+u=8. & (3)\\ z+u+x=9. & (4)\end{cases}$$

Adding, $3u+3x+3y+3z=30.$

Whence, $u+x+y+z=10.$ (5)

Subtracting (2) from (5), $u=3.$

Subtracting (3) from (5), $x=2.$

Subtracting (4) from (5), $y=1.$

Subtracting (1) from (5), $z=4.$

EXERCISE 70

Solve the following:

1. $\begin{cases} 4x-3y= 1. \\ 4y-3z=-15. \\ 4z-3x= 10. \end{cases}$

2. $\begin{cases} 4x-5y-6z= 22. \\ x-y+z=-6. \\ 9x+z= 22. \end{cases}$

3. $\begin{cases} 3x+y-z= 14. \\ x+3y-z= 16. \\ x+y-3z=-10. \end{cases}$

4. $\begin{cases} g+h-k=24. \\ 4g+3h-k=61. \\ 6g-5h-k=11. \end{cases}$

5. $\begin{cases} 3x+5y= 1. \\ 9x+5z=-7. \\ 9y+3z= 2. \end{cases}$

6. $\begin{cases} 5x-y+4z=-5. \\ 3x+5y+6z=-20. \\ x+3y-8z=-27. \end{cases}$

7. $\begin{cases} 2x-5y=-26. \\ 7x+6z=-33. \\ \dfrac{3}{y-4}=\dfrac{4}{z+2}. \end{cases}$

8. $\begin{cases} 2x+4y-z=-2. \\ 18x-8y+4z=-25. \\ 10x+4y-9z=-30. \end{cases}$

9. $\begin{cases} 3p+4q+5r=10. \\ 4p-5q-3r=25. \\ 5p-3q-4r=21. \end{cases}$

10. $\begin{cases} 4x-11y-5z= 9. \\ 8x+4y-z=11. \\ 16x+7y+6z=64. \end{cases}$

11. $\begin{cases} 8x+4y+3z=-52. \\ 5x-y+12z=-52. \\ 9x+7y-6z=-36. \end{cases}$

12. $\begin{cases} 6x-y+3z= 42. \\ 10x-5y-z= 2. \\ 6x-17y+4z=-46. \end{cases}$

13. $\begin{cases} 2x+5y+3z=-7. \\ 2y-4z=2-3x. \\ 5x+9y=5+7z. \end{cases}$

14. $\begin{cases} \dfrac{5}{x}-\dfrac{8}{y}=-3. \\ \dfrac{8}{y}-\dfrac{3}{z}=1. \\ \dfrac{25}{z}+\dfrac{7}{3x}=2. \end{cases}$

15. $\begin{cases} \dfrac{2}{3x}+\dfrac{1}{y}=-\dfrac{3}{10}. \\ \dfrac{3}{4y}-\dfrac{1}{z}= \dfrac{7}{30}. \\ \dfrac{4}{5z}+\dfrac{1}{x}= \dfrac{1}{12}. \end{cases}$

16. $\begin{cases} ax+by=\dfrac{a^3+b^3}{abc}. \\ by+cz=\dfrac{b^3+c^3}{abc}. \\ cz+ax=\dfrac{c^3+a^3}{abc}. \end{cases}$

17. $\begin{cases} 4e - 12t - 20w = 9. \\ 8e - 6t + 10w = 5. \\ 12e - 18t - 5w = 13. \end{cases}$

18. $\begin{cases} u - x + y = 15. \\ x - y + z = -12. \\ y - z + u = 13. \\ z - u + x = -14. \end{cases}$

19. $\begin{cases} \frac{1}{x} + \frac{1}{y} - \frac{1}{z} = m. \\ \frac{1}{y} + \frac{1}{z} - \frac{1}{x} = n. \\ \frac{1}{z} + \frac{1}{x} - \frac{1}{y} = p. \end{cases}$

20. $\begin{cases} ax - by = a^3 - ab^2. \\ ay - bz = a^2b - b^3. \\ az - bx = ab^2 - a^2b. \end{cases}$

21. $\begin{cases} 3u + x = -5. \\ 4x - y = 21. \\ 5y + z = -19. \\ 6z - u = 39. \end{cases}$

22. $\begin{cases} x - \frac{y}{3} + \frac{z}{4} = -3. \\ y - \frac{z}{3} + \frac{x}{4} = \frac{31}{12}. \\ z - \frac{x}{3} + \frac{y}{4} = \frac{21}{2}. \end{cases}$

23. $\begin{cases} \frac{x-y}{3} - \frac{y-z}{4} = \frac{7}{3}. \\ \frac{y-z}{3} + \frac{z+x}{5} = -\frac{13}{15}. \\ \frac{z+x}{2} - \frac{x-y}{5} = \frac{43}{10}. \end{cases}$

24. $\begin{cases} \frac{1}{x} + \frac{1}{y} = a. \\ \frac{1}{y} + \frac{1}{z} = b. \\ \frac{1}{z} + \frac{1}{x} = c. \end{cases}$

25. $\begin{cases} \frac{b}{x} + \frac{a}{y} = c. \\ \frac{a}{z} + \frac{c}{x} = b. \\ \frac{c}{y} + \frac{b}{z} = a. \end{cases}$

26. $\begin{cases} \frac{x}{b-c} + \frac{y}{c-a} = a + b + 2c. \\ \frac{y}{c-a} + \frac{z}{a-b} = b + c + 2a. \\ \frac{z}{a-b} + \frac{x}{b-c} = c + a + 2b. \end{cases}$

27. $\begin{cases} \frac{3}{x+y} + \frac{4}{x-z} = 2. \\ \frac{6}{x+y} - \frac{5}{y-z} = 1. \\ \frac{4}{x-z} + \frac{5}{y-z} = 2. \end{cases}$

28. $\begin{cases} 8x + 9y + 15z = -29. \\ 17x - 10y + 13z = -12. \\ 11x - 15y + 7z = 15. \end{cases}$

29. $\begin{cases} u + 3x - 2y - z = -3. \\ 2u - x - y + 3z = 23. \\ u + x + 3y - 2z = -12. \\ 3u - 2x + y + z = 22. \end{cases}$

30. $\begin{cases} x+y+z=0. \\ (b+c)x+(c+a)y+(a+b)z=0. \\ bcx+cay+abz=1. \end{cases}$

31. $\begin{cases} \frac{d}{3}-\frac{v}{4}+\frac{s}{6}=-9. \\ \frac{d}{4}+\frac{v}{6}-\frac{s}{3}=\ 28. \\ \frac{d}{6}+\frac{v}{3}+\frac{s}{4}=\ 4. \end{cases}$

32. $\begin{cases} \frac{6x+5y}{3}-\frac{y-4z}{5}=-\frac{14}{5}. \\ \frac{z-5x}{6}+\frac{7x+3y}{9}=\ \frac{4}{9}. \\ \frac{4y+6z}{3}+\frac{3z+x}{4}=-\frac{15}{2}. \end{cases}$

PROBLEMS INVOLVING SIMULTANEOUS LINEAR EQUATIONS WITH TWO OR MORE UNKNOWN NUMBERS

175. In solving problems where two or more letters are used to represent unknown numbers, we must obtain from the conditions of the problem *as many independent equations* (§ 164) *as there are unknown numbers to be determined.*

1. Divide 81 into two parts such that three-fifths the greater shall exceed five-ninths the less by 7.

Let $x=$ the greater part,

and $y=$ the less.

By the conditions, $x+y=81,$ (1)

and $\frac{3x}{5}=\frac{5y}{9}+7.$ (2)

Solving (1) and (2), $x=45,\ y=36.$

2. If 3 be added to both numerator and denominator of a fraction, its value is $\frac{2}{3}$; and if 2 be subtracted from both numerator and denominator, its value is $\frac{1}{2}$; find the fraction.

Let $n=$ the numerator,

and $d=$ the denominator.

By the conditions, $\frac{n+3}{d+3}=\frac{2}{3},$

and $\frac{n-2}{d-2}=\frac{1}{2}.$

Solving these equations, $n=7,\ d=12$; then, the fraction is $\frac{7}{12}.$

3. A sum of money was divided equally between a certain number of persons. Had there been 3 more, each would have received \$1 less; had there been 6 fewer, each would have received \$5 more. How many persons were there, and how much did each receive?

Let $x =$ the number of persons,

and $y =$ the number of dollars received by each.

Then, $xy =$ the number of dollars divided.

Since the sum of money could be divided between $x + 3$ persons, each of whom would receive $y - 1$ dollars, and between $x - 6$ persons, each of whom would receive $y + 5$ dollars, $(x + 3)(y - 1)$ and $(x - 6)(y + 5)$ also represent the number of dollars divided.

Then, $(x + 3)(y - 1) = xy,$

and $(x - 6)(y + 5) = xy.$

Solving these equations, $x = 12,\ y = 5.$

4. The sum of the three digits of a number is 13. If the number, decreased by 8, be divided by the sum of its second and third digits, the quotient is 25; and if 99 be added to the number, the digits will be reversed. Find the number.

Let $x =$ the first digit,

$y =$ the second,

and $z =$ the third.

Then, $100x + 10y + z =$ the number,

and $100z + 10y + x =$ the number with its digits reversed.

By the conditions of the problem,

$$x + y + z = 13,$$

$$\frac{100x + 10y + z - 8}{y + z} = 25,$$

and $$100x + 10y + z + 99 = 100z + 10y + x.$$

Solving these equations, $x = 2,\ y = 8,\ z = 3$; and the number is 283.

5. A crew can row 10 miles in 50 minutes down stream, and 12 miles in $1\frac{1}{2}$ hours against the stream. Find the rate in miles per hour of the current, and of the crew in still water.

Let $x =$ number of miles an hour of the crew in still water,

and $y =$ number of miles an hour of the current.

Then, $x + y =$ number of miles an hour of the crew down stream,

and $x - y =$ number of miles an hour of the crew up stream.

The number of miles an hour rowed by the crew is equal to the distance in miles divided by the time in hours.

Then,
$$x + y = 10 \div \frac{5}{6} = 12,$$

and
$$x - y = 12 \div \frac{3}{2} = 8.$$

Solving these equations, $5 = 10,\ y = 2.$

6. A train running from A to B meets with an accident which causes its speed to be reduced to one-third of what it was before, and it is in consequence 5 hours late. If the accident had happened 60 miles nearer B, the train would have been only 1 hour late. Find the rate of the train before the accident, and the distance to B from the point of detention.

Let $3x =$ the number of miles an hour of the train before the accident.

Then, $x =$ the number of miles an hour after the accident.

Let $y =$ the number of miles to B from the point of detention.

The train would have done the last y miles of its journey in $\frac{y}{3x}$ hours; but owing to the accident, it does the distance in $\frac{y}{x}$ hours.

Then,
$$\frac{y}{x} = \frac{y}{3x} + 5. \qquad (1)$$

If the accident had occurred 60 miles nearer B, the distance to B from the point of detention would have been $y - 60$ miles.

Had there been no accident, the train would have done this in $\frac{y-60}{3x}$ hours, and the accident would have made the time $\frac{y-60}{x}$ hours.

Then,
$$\frac{y-60}{x} = \frac{y-60}{3x} + 1. \qquad (2)$$

Subtracting (2) from (1), $\frac{60}{x} = \frac{60}{3x} + 4$, or $\frac{40}{x} = 4$; whence, $x = 10$.

Then, the rate of the train before the accident was 30 miles an hour.

Substituting in (1), $\frac{y}{10} = \frac{y}{30} + 5$, or $\frac{y}{15} = 5$; whence, $y = 75$.

EXERCISE 71

1. Divide 79 into two parts such that three-sevenths the less shall be less by 56 than four-thirds the greater.

2. If the numerator of a fraction be increased by 4, the value of the fraction is $\frac{4}{5}$; while if the denominator is decreased by 3, the value of the fraction is $\frac{2}{3}$. Find the fraction.

3. The sum of the two digits of a number is 14; and if 36 be added to the number, the digits will be reversed. Find the number.

4. A's age is $\frac{6}{5}$ of B's, and 15 years ago his age was $\frac{13}{10}$ of B's. Find their ages.

5. If the two digits of a number be reversed, the quotient of the number thus formed, increased by 101, by the original number is 2; and the sum of the digits exceeds twice the excess of the tens' digit over the units' digit by 5. Find the number.

6. If 3 be added to the numerator of a fraction, and 7 subtracted from the denominator, its value is $\frac{6}{7}$; and if 1 be subtracted from the numerator, and 7 added to the denominator, its value is $\frac{2}{5}$. Find the fraction.

7. A's age is twice the sum of the ages of B and C; two years ago, A was 4 times as old as B, and four years ago, A was 6 times as old as C. Find their ages.

8. If the greater of two numbers be divided by the less, the quotient is 1, and the remainder 6. And if the greater, increased by 14, be divided by the less, diminished by 4, the quotient is 5, and the remainder 4. Find the numbers.

9. If 8 yards of silk and 12 yards of woolen cost \$ 27, and 12 yards of silk and 8 yards of woolen cost \$ 28, find the price per yard of the silk and of the woolen.

10. Find two numbers such that one shall be n times as much greater than a as the other is less than a; and the quotient of their sum by their difference equal to b.

11. A certain number of two digits exceeds three times the sum of its digits by 4. If the digits be reversed, the sum of the resulting number and the given number exceeds three times the given number by 2. Find the number.

12. The sum of the three digits of a number is 16; the digit in the tens' place exceeds that in the hundreds' place by 4; and if 297 be added to the number, the digits will be reversed. Find the number.

13. A rectangular field has the same area as another which is 6 rods longer and 2 rods narrower, and also the same area as a third which is 3 rods shorter and 2 rods wider. Find its dimensions.

14. Find three numbers such that the first with one-half the second and one-third the third shall equal 29; the second with one-third the first and one-fourth the third shall equal 28; and the third with one-half the first and one-third the second shall equal 36.

15. The circumference of the large wheel of a carriage is 55 inches more than that of the small wheel. The former makes as many revolutions in going 250 feet as the latter does in going 140 feet. Find the number of inches in the circumference of each wheel.

16. If the digits of a number of three figures be reversed, the sum of the number thus formed and the original number is 1615; the sum of the digits is 20, and if 99 be added to the number, the digits will be reversed. Find the number.

17. A train leaves *A* for *B*, 112 miles distant, at 9 A.M., and one hour later a train leaves *B* for *A*; they met at 12 noon. If the second train had started at 9 A.M., and the first at 9.50 A.M., they would also have met at noon. Find their rates.

18. A boy has $1.50 with which he wishes to buy two kinds of note-books. If he asks for 14 of the first kind, and 11 of the second, he will require 6 cents more; and if he asks for 11 of the first kind, and 14 of the second, he will have 6 cents over. How much does each kind cost?

19. A man invests \$10,000, part at $4\frac{1}{4}\%$, and the rest at $3\frac{1}{2}\%$. He finds that six years' interest on the first investment exceeds five years' interest on the second by \$658. How much does he invest at each rate?

20. A man buys apples, some at 2 for 3 cents, and others at 3 for 2 cents, spending in all 80 cents. If he had bought $\frac{5}{4}$ as many of the first kind, and $\frac{6}{5}$ as many of the second, he would have spent 99 cents. How many of each kind did he buy?

21. An annual income of \$800 is obtained in part from money invested at $3\frac{1}{2}\%$, and in part from money invested at 3%. If the amount invested at the first rate were invested at 3%, and the amount invested at the second rate were invested at $3\frac{1}{2}\%$, the annual income would be \$825. How much is invested at each rate?

22. A tank containing 864 gallons can be filled by two pipes, A and B. After the pipes have been open together for 9 minutes, the pipe A is closed, and B finishes the work of filling in $15\frac{3}{4}$ minutes. If 15 minutes had elapsed before the pipe A was closed, B would have finished in $2\frac{1}{4}$ minutes. How many gallons does each pipe fill in one minute?

23. The contents of one barrel is $\frac{5}{8}$ wine, and of another $\frac{8}{9}$ wine. How many gallons must be taken from each to fill a barrel whose capacity is 24 gallons, so that the mixture may be $\frac{7}{8}$ wine?

24. A boy spends his money for oranges. Had he bought m more, each would have cost a cents less; if n fewer, each would have cost b cents more. How many did he buy, and at what price?

25. A vessel contains a mixture of wine and water. If 50 gallons of wine are added, there is $\frac{7}{8}$ as much wine as water; if 50 gallons of water are added, there is 4 times as much water as wine. Find the number of gallons of wine and water at first.

26. A man buys 15 bottles of sherry, and 20 bottles of claret, for \$38. If the sherry had cost $\frac{5}{4}$ as much, and the claret $\frac{4}{5}$ as much, the wine would have cost \$38.50. Find the cost per bottle of the sherry, and of the claret.

27. If a field were made a feet longer, and b feet wider, its area would be increased by m square feet; but if its length were made c feet less, and its width d feet less, its area would be decreased by n square feet. Find its dimensions.

28. If the numerator of a fraction be increased by a, and the denominator by b, the value of the fraction is $\frac{m}{n}$; and if the numerator be decreased by c, and the denominator by d, the value of the fraction is $\frac{n}{m}$. Find the numerator and denominator.

29. A certain number equals 59 times the sum of its three digits. The sum of the digits exceeds twice the tens' digit by 3; and the sum of the hundreds' and tens' digits exceeds twice the units' digit by 6. Find the number.

30. A piece of work can be done by A and B in $4\frac{4}{5}$ hours, by B and C in $2\frac{2}{3}$ hours, and by A and C in 3 hours. In how many hours can each alone do the work?

31. The numerator of a fraction has the same two digits as the denominator, but in reversed order; the denominator exceeds the numerator by 9, and if 1 be added to the numerator the value of the fraction is $\frac{3}{4}$. Find the fraction.

32. A man walks from one place to another in $5\frac{1}{2}$ hours. If he had walked $\frac{1}{4}$ of a mile an hour faster, the walk would have taken $36\frac{2}{3}$ fewer minutes. How many miles did he walk, and at what rate?

33. A man invests a certain sum of money at a certain rate of interest. If the principal had been \$1200 greater, and the rate 1% greater, his income would have been increased by \$118. If the principal had been \$3200 greater, and the rate 2% greater, his income would have been increased by \$312. What sum did he invest, and at what rate?

34. A sum of money at simple interest amounted to \$ 1868.40 in 7 years, and to \$ 2174.40 in 12 years. Find the principal and the rate.

35. A and B together can do a piece of work in $3\frac{3}{5}$ hours. If A works $\frac{2}{3}$ as fast, and B $\frac{3}{2}$ as fast, they can do it in the same time. In how many hours can each alone do the work?

36. Two men together can do a piece of work in 30 hours; they can also do it if the first man works $25\frac{1}{5}$ hours, and the second $32\frac{4}{5}$ hours. In how many hours can each alone do the work?

37. A crew row $16\frac{1}{2}$ miles up stream and 18 miles down stream in 9 hours. They then row 21 miles up stream and $19\frac{1}{2}$ miles down stream in 11 hours. Find the rate in miles an hour of the stream, and of the crew in still water.

38. A train travels from A to B, 228 miles, and another from B to A. If the trains start at the same time, they will meet $3\frac{4}{5}$ hours after. If the first train starts 3 hours before the second, they will meet 2 hours after the second train starts. Find the rates of the trains.

39. A man has quarter-dollars, dimes, and half-dimes to the value of \$ 1.40, and has in all 12 coins. If he replaces the quarters by dimes, and the dimes by quarters, the value of the coins would be \$ 1.55. How many has he of each?

40. The middle digit of a number of three figures is one-half the sum of the other two digits. If the number be divided by the sum of its digits, the quotient is 20, and the remainder 9; and if 594 be added to the number, the digits will be reversed. Find the number.

41. A certain number of workmen receive the same wages, and receive together a certain sum. If there had been 9 more men, and each had received 30 cents less, the total received would have been increased by \$ 12.30. Had there been 8 fewer men, and each had received 40 cents more, the total received would have been decreased by \$ 13.20. How many men were there, and how much did each receive?

42. A merchant has three casks of wine, containing together 66 gallons. He pours from the first into the second and third as much as each of them contains; he then pours from the second into the first and third as much as each of them then contains. There is now 8 times as much in the third cask as in the second, and twice as much in the first as in the second. How many gallons did each have at first?

43. In a meeting of 600 persons, a measure is defeated by a certain majority. It is afterwards successful by double this majority, and the number of persons voting for it is $\frac{5}{4}$ as great as the number voting against on the former occasion. How many voted for, and how many against, the measure on the former occasion?

44. I bought apples at 3 for 5 cents, and oranges at 2 for 5 cents, spending in all $ 1.70. I sold three-fourths of the apples and one-half of the oranges for $ 1.10, and made a profit of 5 cents on the latter transaction. How many did I buy of each?

45. A gives to B and C as much money as each of them has; B then gives to A and C as much money as each of them then has; C then gives to A and B as much money as each of them then has. Each has now $ 8. How much had each at first?

46. A has one-half as many dimes as dollars, and B eight-sevenths as many dimes as dollars. They have together 3 more dollars than dimes, and B's money is 60 cents less than A's. How much money has each?

47. A man buys a certain number of $ 100 railway shares, when at a certain rate per cent discount, for $ 1050; and when at a rate per cent premium twice as great, sells one-half of them for $ 1200. How many shares did he buy, and at what cost?

48. A and B can do a piece of work in $\frac{35}{12}$ hours, A and C in $\frac{14}{9}$ hours, A and D in $\frac{21}{10}$ hours, and B and C in $\frac{10}{7}$ hours. How many hours will it take each alone to do the work?

49. A and B run a race of 280 feet. The first heat, A gives B a start of 70 feet, and neither wins the race. The second heat, A gives B a start of 35 feet, and beats him by $6\frac{2}{3}$ seconds. How many feet can each run in a second?

50. A, B, C, and D play at cards. After B has won one-half of A's money, C one-third of B's, D one-fourth of C's, and A one-fifth of D's, they have each \$10, except B, who has \$16. How much had each at first?

51. The sum of the four digits of a number is 14. The sum of the last three digits exceeds twice the first by 2. Twice the sum of the second and third digits exceeds 3 times the sum of the first and fourth by 3. If 2727 be subtracted from the number, the digits will be reversed. Find the number.

52. A and B run a race of 210 yards. The first heat, A gives B a start of 8 seconds, and beats him by 20 yards. The second heat, A gives B a start of 70 yards, and is beaten by 2 seconds. How many yards can each run in a second?

53. A sum of money consists of half-dollars, dimes, and half-dimes. Its value is as many dimes as there are pieces of money; and its value is also as many half-dollars as there are dimes less 1. The number of dimes is 5 more than the number of half-dollars. Find the number of each coin.

54. The fore-wheel of a carriage makes a revolutions more than the hind-wheel in travelling b feet. If the circumference of the fore-wheel were m times as great, and the circumference of the hind-wheel n times as great, the fore-wheel would make c revolutions more than the hind-wheel in travelling d feet. Find the circumference of each wheel.

55. A train running from A to B meets with an accident. It then proceeds at a rate one-nth of its former rate, and arrives at B b hours late. Had the accident occurred c miles nearer B, the train would have been d hours late. Find the rate of the train before the accident, and the distance to B from the point of detention.

56. A man buys 60 shares of stock, each having the par value \$100, part paying dividends at the rate of $3\frac{3}{4}\%$, and the remainder at the rate of $4\frac{1}{2}\%$. If the first part had paid dividends at the rate of $4\frac{1}{2}\%$, and the other at the rate of $3\frac{3}{4}\%$, the total annual income would have been \$12 less. How many shares of each kind did he buy?

176. Interpretation of Solutions.

1. The length of a field is 10 rods, and its breadth 8 rods; how many rods must be added to the breadth so that the area may be 60 square rods?

Let $x =$ number of rods to be added.

By the conditions, $10(8+x)=60$.

Then, $80+10x=60$, or $x=-2$.

This signifies that 2 rods must be *subtracted* from the breadth in order that the area may be 60 square rods. (Compare § 16.)

If we should modify the problem so as to read:

"The length of a field is 10 rods, and its breadth 8 rods; how many rods must be *subtracted* from the breadth so that the area may be 60 square rods?"

and let x denote the number of rods to be subtracted, we should find $x=2$.

A negative result sometimes indicates that the problem is impossible.

2. If 11 times the number of persons in a certain house, increased by 18, be divided by 4, the result equals twice the number increased by 3; find the number.

Let $x =$ the number.

By the conditions, $\frac{11x+18}{4}=2x+3$.

Whence, $11x+18=8x+12$, and $x=-2$.

The negative result shows that the problem is impossible.

A problem may also be impossible when the solution is fractional.

3. A man has two kinds of money: dimes and cents. The total number of coins is 23, and their value 37 cents. How many has he of each?

Let $x =$ number of dimes.

Then, $23-x =$ number of cents.

The x dimes are worth $10x$ cents.

Then, by the conditions, $10x + 23 - x = 37$; and $x = \frac{14}{9}$.

The fractional result shows that the problem is impossible.

EXERCISE 72

Interpret the solutions of the following:

1. If the length of a field is 12 rods, and its width 9 rods, how many rods must be subtracted from the width so that the area may be 144 square rods?

2. A is 44 years of age, and B 12 years; how many years ago was A 3 times as old as B?

3. The number of apple and pear trees in an orchard is 23; and 7 times the number of apple trees plus twice the number of pear trees equals 82. How many are there of each kind?

4. The number of silver coins in a purse exceeds the number of gold coins by 3, and 5 times the number of silver coins exceeds 3 times the number of gold coins by 3. How many are there of each kind?

5. A's assets are double those of B. When A has gained \$250, and B \$170, A's assets are 5 times those of B. Find the assets of each.

6. A cistern has two pipes. When both are open, it is filled in $7\frac{1}{2}$ hours; and the first pipe alone can fill it in 3 hours. How many hours does the second pipe take to fill it?

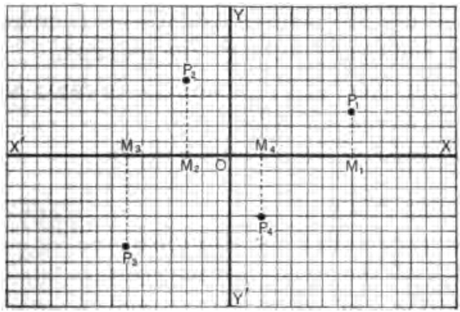
7. The numerator of a fraction is 4 times the denominator; and if the numerator be diminished by 9, and the denominator by 15, the value of the fraction is $\frac{3}{8}$. Find the fraction.

8. A and B are travelling due east at the rates of $4\frac{1}{2}$ and $3\frac{1}{2}$ miles an hour, respectively. At noon A is 5 miles due east of B. How many miles to the east of A's position at noon will he overtake B?

9. A has \$720, and B \$300. After A has gained a certain sum, and B has gained two-thirds this sum, A has 3 times as much money as B. How much did each gain?

XIII. GRAPHICAL REPRESENTATION

177. Rectangular Co-ordinates of a Point.



Let XX' and YY' be straight lines intersecting at right angles at O; let P_1 be any point in the plane of XX' and YY', and draw line P_1M_1 perpendicular to XX'.

Then, OM_1 and M_1P_1 are called the *rectangular co-ordinates*, or simply the *co-ordinates*, of P_1; OM_1 is called the *abscissa*, and M_1P_1 the *ordinate*.

178. It is understood, in the definitions of § 177, that abscissas measured to the *right* of O are *positive*, and to the *left*, *negative*; also, that ordinates measured *upwards* from XX' are *positive*, and *downwards*, *negative*.

Thus, let P_2 be to the left of YY', and above XX', and P_3 and P_4 below XX', respectively to the left and right of YY', and draw lines P_2M_2, P_3M_3, and P_4M_4 perpendicular to XX'.

Let $\quad OM_1 = 8, \quad M_2O = 3, \quad M_3O = 7, \quad OM_4 = 2,$

$\qquad M_1P_1 = 3, \quad M_2P_2 = 5, \quad P_3M_3 = 6, \quad P_4M_4 = 4.$

Then, the abscissa of P_1 is $+8$, and its ordinate $+3$;
the abscissa of P_2 is -3, and its ordinate $+5$;
the abscissa of P_3 is -7, and its ordinate -6;
the abscissa of P_4 is $+2$, and its ordinate -4.

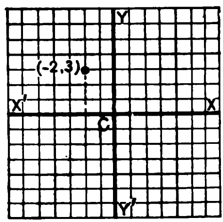
179. The lines of reference, XX' and YY', are called the *axis* of X, and *axis* of Y, respectively; and O the *origin*.

We express the fact that the abscissa of a point is b, and its ordinate a, by saying that, for the point in question, $x = b$ and $y = a$; or, more concisely, we speak of the point as the point (b, a); where the first term in parentheses is understood to be the abscissa, and the second term the ordinate.

If a point lies upon XX', its ordinate is zero; and if it lies upon YY', its abscissa is zero.

The co-ordinates of the origin are (0, 0).

180. Plotting Points.



To *plot* a point when its co-ordinates are given, lay off the abscissa to the right or left of O, according as it is $+$ or $-$, and then draw a perpendicular, equal in length to the ordinate, above or below XX', according as the ordinate is $+$ or $-$.

Thus, to plot the point $(-2, 3)$, lay off 2 units to the left of O upon XX', and then erect a perpendicular 3 units in length above XX'.

EXERCISE 73

Plot the following points:

1. $(1, 4)$.
2. $(2, -2)$.
3. $(-3, 6)$.
4. $(-2, -4)$.
5. $(3, 1)$.
6. $(-4, -3)$.
7. $(-1, 2)$.
8. $(4, -6)$.
9. $(7, 3)$.
10. $(-6, 1)$.
11. $(5, 0)$.
12. $(0, 4)$.
13. $(-2, 0)$.
14. $(0, -3)$.

GRAPH OF A LINEAR EQUATION INVOLVING TWO UNKNOWN NUMBERS

181. Consider the equation $y = x + 2$.

If we give any numerical value to x, we may, by aid of the relation $y = x + 2$, calculate a corresponding value for y.

If $x=0,\quad y=2.$

If $x=1,\quad y=3.$

If $x=2,\quad y=4.$

If $x=4,\quad y=6.$

If $x=6,\quad y=8.$

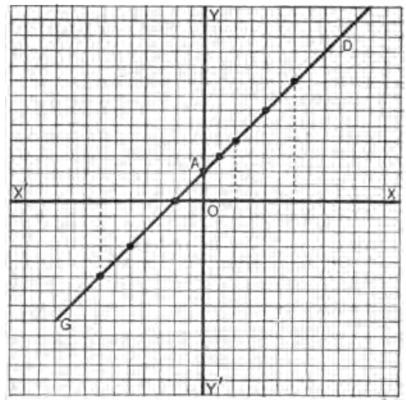
If $x=-1,\quad y=0.$

If $x=-3,\quad y=-1.$

If $x=-5,\quad y=-3.$

If $x=-7,\quad y=-5;$

etc.



Scale: $\frac{1}{16}$ inch = 1 unit.

Now let these be regarded as the co-ordinates of points; and let the points be plotted, as explained in § 180.

Thus, to plot the point A, lay off 2 units above O upon YY'.

The points will be found to lie on a certain line, GD, which is called the **Graph** of the given equation.

By assuming fractional values for x, we may obtain intermediate points of the graph.

EXERCISE 74

Find by the above method the graphs of the following equations:

1. $y=2x+3.$
2. $y=-3x-4.$
3. $4y+x=6.$
4. $3y-2x=-12.$
5. $y=5x.$
6. $3x+2y=0.$

182. We shall always find (and it can be proved) that a linear equation, involving two unknown numbers, has a *straight line* for a graph.

Then, since a straight line is determined by any two of its points, it is sufficient, when finding the graph of a linear equation involving two unknown numbers, to find two of its points, and draw a straight line through them.

The points most easily determined are those in which the graph intersects the axes.

For all points on OX, $y=0$; hence, to find where the graph cuts OX, put $y=0$, and calculate the value of x.

To find where the graph cuts OY, put $x=0$, and calculate the value of y.

Ex. Plot the graph of $2x+3y=-7$.

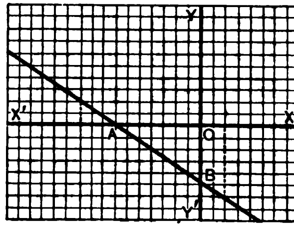
Put $y=0$; then $2x=-7$, and $x=-\frac{7}{2}$.

Then plot A on OX', $\frac{7}{2}$ units to the left of O.

Put $x=0$; then $3y=-7$ and $y=-\frac{7}{3}$.

Then plot B on OY', $\frac{7}{3}$ units below O.

Draw the straight line AB; this is the required graph.



Scale: $\frac{1}{8}$ inch = 1 unit.

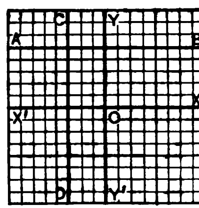
The above method, cannot, of course, be used for a straight line passing through the origin, nor for the equations of § 183.

183. Consider the equation $y=5$.

This means that every point in the graph has its ordinate equal to 5.

Then the graph is the straight line AB, parallel to XX', and 5 units above it.

In like manner, the graph of $x=-3$ is the straight line CD, parallel to YY', and 3 units to the left of it.



Scale: $\frac{1}{16}$ inch = 1 unit.

The graph of $y=0$ is the axis of X, and the graph of $x=0$ is the axis of Y.

EXERCISE 75

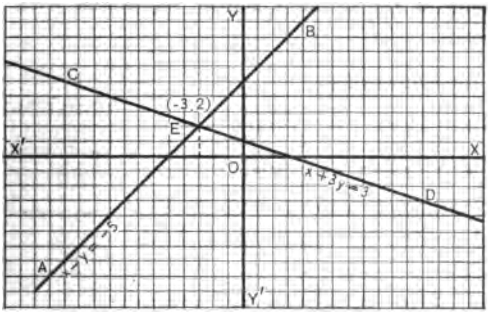
Plot the graphs of the following equations:

1. $3x+2y=6$.
2. $x-4y=4$.
3. $x=2$.
4. $y=-4$.
5. $16x-27y=-72$
6. $8x+15y=-6$.

INTERSECTIONS OF GRAPHS

184. Consider the equations

$$\begin{cases} x - y = -5. & (AB) \\ x + 3y = 3. & (CD) \end{cases}$$



Let AB be the graph of $x - y = -5$, and CD the graph of $x + 3y = 3$.

Let AB and CD intersect at E.

Since E lies on each graph, its co-ordinates must satisfy both given equations; hence, to find the co-ordinates of E, we solve the given equations.

In this case the solution is $x = -3$, $y = 2$; and it may be verified in the figure that these are the co-ordinates of E.

We then have the following important principle:

If the graphs of two linear equations, with two unknown numbers, intersect, the co-ordinates of the point of intersection form a solution of the equations represented by the graphs.

EXERCISE 76

Verify the principle of § 184 in the following equations:

1. $\begin{cases} 4x + 5y = 24. \\ 3x - 2y = -5. \end{cases}$

2. $\begin{cases} 3x + 7y = 5. \\ 8x + 3y = -18. \end{cases}$

3. $\begin{cases} 5x - 4y = 0. \\ 7x + 6y = -29. \end{cases}$

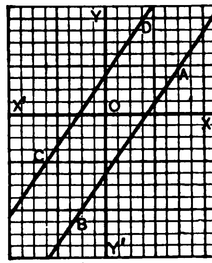
4. $\begin{cases} 9x + 14y = -25. \\ 3x - 4y = 22. \end{cases}$

As additional examples, the pupil might verify graphically the solutions of Exs. 3, 8, 11, and 12, Exercise 65, and of Exs. 7, 8, 9, and 16, Exercise 66.

185. Graphs of Inconsistent Linear Equations with Two Unknown Numbers.

Consider the equations

$$\begin{cases} 3x - 2y = 5. & (AB) \\ 6x - 4y = -7. & (CD) \end{cases}$$



Scale: $\frac{1}{8}$ inch = 1 unit.

The first equation can be put in the form $6x - 4y = 10$, by multiplying both members by 2.

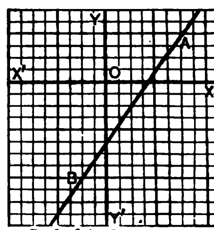
Then, the given equations are *inconsistent* (§ 165), and it is impossible to find any values of x and y which satisfy both equations.

We shall always find that two inconsistent equations, with two unknown numbers, are represented by *parallel graphs;* for if the graphs could intersect at any point, the co-ordinates of this point would be a solution of the given equations (§ 184).

186. Graphs of Indeterminate Linear Equations with Two Unknown Numbers.

Consider the equations

$$\begin{cases} 3x - 2y = 5. \\ 6x - 4y = 10. \end{cases}$$



Scale $\frac{1}{8}$ inch = 1 unit.

The first equation can be put in the form of the second, by multiplying both members by 2, and the graphs coincide.

The given equations are not *independent* (§ 164); in any similar case, we shall find that the graphs are coincident.

EXERCISE 77

Verify the principles of §§ 185 and 186 in the following equations:

1. $\begin{cases} 3x+4y=12. \\ 3x+4y=-12. \end{cases}$

2. $\begin{cases} 2x-5y=0. \\ 6x-15y=30. \end{cases}$

3. $\begin{cases} 2x-7y=14. \\ 4x-14y=28. \end{cases}$

4. $\begin{cases} 5x+6y=15. \\ 15x+18y=45. \end{cases}$

187. Graphical Representation of Linear Expressions involving One Unknown Number.

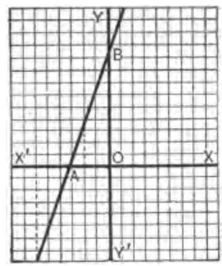
Consider the expression $3x+5$.

Put $y=3x+5$; and let the graph of this equation be found as in § 183.

Putting $y=0$, $x=-\frac{5}{3}$; then the graph cuts XX' $\frac{5}{3}$ units to the left of O.

Putting $x=0$, $y=5$; then the graph cuts YY' 5 units above O.

The graph is the straight line AB.



Scale $\frac{1}{8}$ inch = 1 unit.

188. Graphical Representation of Roots of Equations (§ 81).

In order to find the abscissa of the point A (§ 187), where the graph of $3x+5$ intersects XX', we solve the equation $3x+5=0$ (§ 182).

That is, the abscissa of A is a root of the equation $3x+5=0$.

Hence, *the abscissa of the point in which the graph of the first member of any linear equation, with one unknown number, intersects XX', is the root of the equation.*

EXERCISE 78

Plot the graphs of the first members of the following equations, and in each case verify the principle of § 188:

1. $2x+7=0$.

2. $5x-4=0$.

XIV. INEQUALITIES

189. The **Signs of Inequality,** $>$ and $<$, are read "*is greater than*" and "*is less than*," respectively.

Thus, $a > b$ is read "a is greater than b"; $a < b$ is read "a is less than b."

190. One number is said to be *greater* than another when the remainder obtained by subtracting the second from the first is a *positive* number.

One number is said to be *less* than another when the remainder obtained by subtracting the second from the first is a *negative* number.

Thus, if $a - b$ is a positive number, $a > b$; and if $a - b$ is a negative number, $a < b$.

191. An **Inequality** is a statement that one of two expressions is greater or less than another.

The *First Member* of an inequality is the expression to the left of the sign of inequality, and the *Second Member* is the expression to the right of that sign.

Any term of either member of an inequality is called a *term* of the inequality.

Two or more inequalities are said to *subsist in the same sense* when the first member is the greater or the less in both.

Thus, $a > b$ and $c > d$ subsist in the same sense.

PROPERTIES OF INEQUALITIES

192. *An inequality will continue in the same sense after the same number has been added to, or subtracted from, both members.*

For consider the inequality $a > b$.

By § 190, $a - b$ is a positive number.

Hence, each of the numbers

$$(a+c)-(b+c), \text{ and } (a-c)-(b-c)$$

is positive, since each is equal to $a-b$.

Therefore, $a+c>b+c$, and $a-c>b-c$. (§ 190)

193. It follows from § 192 that *a term may be transposed from one member of an inequality to the other by changing its sign.*

If the same term appears in both members of an inequality, affected with the same sign, it may be cancelled.

194. *If the signs of all the terms of an inequality be changed the sign of inequality must be reversed.*

For consider the inequality $a-b>c-d$.

Transposing every term, $d-c>b-a$. (§ 193)

That is, $b-a<d-c$.

195. *An inequality will continue in the same sense after both members have been multiplied or divided by the same positive number.*

For consider the inequality $a>b$.
By § 190, $a-b$ is a positive number.
Hence, if m is a positive number, each of the numbers

$$m(a-b) \text{ and } \frac{a-b}{m}, \text{ or } ma-mb \text{ and } \frac{a}{m}-\frac{b}{m}, \text{ is positive.}$$

Therefore, $ma>mb$, and $\dfrac{a}{m}>\dfrac{b}{m}$.

196. It follows from §§ 194 and 195 that *if both members of an inequality be multiplied or divided by the same negative number, the sign of inequality must be reversed.*

197. *If any number of inequalities, subsisting in the same sense, be added member to member, the resulting inequality will also subsist in the same sense.*

For consider the inequalities $a > b,\ a' > b',\ a'' > b'',\ \cdots$.

Each of the numbers, $a - b,\ a' - b',\ a'' - b'',\ \cdots$, is positive.

Then, their sum $a - b + a' - b' + a'' - b'' + \cdots$,

or, $$a + a' + a'' + \cdots - (b + b' + b'' + \cdots),$$

is a positive number.

Whence, $$a + a' + a'' + \cdots > b + b' + b'' + \cdots.$$

If two inequalities, subsisting in the same sense, be *subtracted* member from member, the resulting inequality does not necessarily subsist in the same sense.

Thus, if $a > b$ and $a' > b'$, the numbers $a - b$ and $a' - b'$ are positive.

But $(a - b) - (a' - b')$, or its equal, $(a - a') - (b - b')$, may be positive, negative, or zero; and hence $a - a'$ may be greater than, less than, or equal to $b - b'$.

198. If $a > b$ and $a' > b'$, and each of the numbers $a,\ a',\ b,\ b'$, is positive, then
$$aa' > bb'.$$

Since $a' > b'$, and a is positive,
$$aa' > ab' \quad (\S\ 195). \qquad (1)$$

Again, since $a > b$, and b' is positive,
$$ab' > bb'. \qquad (2)$$

From (1) and (2), $$aa' > bb'.$$

199. If we have any number of inequalities subsisting in the same sense, as $a > b,\ a' > b',\ a'' > b'',\ \cdots$, and each of the numbers $a,\ a',\ a'',\ \cdots,\ b,\ b',\ b'',\ \cdots$, is positive, then
$$aa'a'' \cdots > bb'b'' \cdots.$$

For by § 198, $$aa' > bb'.$$

Also, $$a'' > b''.$$

Then by § 198, $$aa'a'' > bb'b''.$$

Continuing the process with the remaining inequalities, we obtain finally
$$aa'a'' \cdots > bb'b'' \cdots.$$

200. Examples.

1. Find the limit of x in the inequality

$$7x-\frac{23}{3}<\frac{2x}{3}+5.$$

Multiplying both members by 3 (§ 195), we have

$$21x-23<2x+15.$$

Transposing (§ 193), and uniting terms,

$$19x<38.$$

Dividing both members by 19 (§ 195),

$$x<2.$$

$\left(\text{This means that, for any value of } x<2,\ 7x-\frac{23}{3}<\frac{2x}{3}+5.\right)$

2. Find the limits of x and y in the following:

$$\begin{cases} 3x+2y>37. & (1)\\ 2x+3y=33. & (2)\end{cases}$$

Multiply (1) by 3,	$9x+6y>111.$
Multiply (2) by 2,	$4x+6y=\ 66.$
Subtracting (§ 192),	$5x>\ 45$, and $x>9$.
Multiply (1) by 2,	$6x+4y>\ 74.$
Multiply (2) by 3,	$6x+9y=\ 99.$
Subtracting,	$-5y>-25.$
Divide both members by	-5, $y<5$ (§ 196).

(This means that any values of x and y which satisfy (2), also satisfy (1), provided x is >9, and $y<5$.)

3. Between what limiting values of x is $x^2-4x<21$?

Transposing 21, we have

$$x^2-4x \text{ is } <21, \text{ if } x^2-4x-21 \text{ is } <0.$$

That is, if $(x+3)(x-7)$ is negative.

Now $(x+3)(x-7)$ is negative if x is between -3 and 7; for if x is <-3, both $x+3$ and $x-7$ are negative, and their product positive; and if x is >7, both $x+3$ and $x-7$ are positive.

Hence, x^2-4x is <21, if x is >-3, and <7.

EXERCISE 79

Find the limits of x in the following:

1. $(4x+5)^2-4<(8x+5)(2x+3)$.
2. $(3x+2)(x+3)-4x>(3x-2)(x-3)+36$.
3. $(x+4)(5x-2)+(2x-3)^2>(3x+4)^2-78$.
4. $(x-3)(x+4)(x-5)<(x+1)(x-2)(x-3)$.
5. $a^2(x-1)<2b^2(2x-1)-ab$, if $a-2b$ is positive.
6. $\dfrac{x-m}{n}+2>\dfrac{x+n}{m}$, if m and n are positive, and $m<n$.

Find the limits of x and y in the following:

7. $\begin{cases} 5x+6y<45. \\ 3x-4y=-11. \end{cases}$
8. $\begin{cases} 7x-4y>41. \\ 3x+7y=35. \end{cases}$

9. Find the limits of x when

$$3x-11<24-11x, \text{ and } 5x+23<20x+3.$$

10. If 6 times a certain positive integer, plus 14, is greater than 13 times the integer, minus 63, and 17 times the integer, minus 23, is greater than 8 times the integer, plus 31, what is the integer?

11. If 7 times the number of houses in a certain village, plus 33, is less than 12 times the number, minus 82, and 9 times the number, minus 43, is less than 5 times the number, plus 61, how many houses are there?

12. A farmer has a number of cows such that 10 times their number, plus 3, is less than 4 times the number, plus 79; and 14 times their number, minus 97, is greater than 6 times the number, minus 5. How many cows has he?

13. Between what limiting values of x is $x^2+3x<4$?

14. Between what limiting values of x is $x^2<8x-15$.

15. Between what limiting values of x is $3x^2+19x<-20$?

201. *If a and b are unequal numbers,*

$$a^2+b^2>2ab.$$

For $(a-b)^2 > 0$; or, $a^2 - 2ab + b^2 > 0$.

Transposing $-2ab$, $a^2 + b^2 > 2ab$.

1. Prove that, if a does not equal 3,

$$(a+2)(a-2) > 6a - 13.$$

By the above principle, if a does not equal 3,

$$a^2 + 9 > 6a.$$

Subtracting 13 from both members,

$$a^2 - 4 > 6a - 13, \text{ or } (a+2)(a-2) > 6a - 13.$$

2. Prove that, if a and b are unequal positive numbers,

$$a^3 + b^3 > a^2b + b^2a.$$

We have, $a^2 + b^2 > 2ab$, or $a^2 - ab + b^2 > ab$.

Multiplying both members by the positive number $a + b$,

$$a^3 + b^3 > a^2b + b^2a.$$

EXERCISE 80

1. Prove that for any value of x, except $\frac{5}{3}$,

$$3x(3x - 10) > -25.$$

2. Prove that for any value of x, except $\frac{7}{2}$,

$$4x(x-5) > 8x - 49.$$

3. Prove that for any values of a and b, if $4a$ does not equal $3b$,

$$(4a + 3b)(4a - 3b) > 6b(4a - 3b).$$

4. Prove that for any values of x and y, if $5x$ does not equal $4y$,

$$5x(5x - 6y) > 2y(5x - 8y).$$

Prove that, if a and b are unequal positive numbers,

5. $a^3b + ab^3 > 2a^2b^2$.

6. $\frac{a}{b} + \frac{b}{a} > 2$.

7. $a^3 + a^2b + ab^2 + b^3 > 2ab(a+b)$.

XV. INVOLUTION

202. **Involution** is the process of raising an expression to any power whose exponent is a positive integer.

We gave in § 96 a rule for raising a monomial to any power whose exponent is a positive integer.

203. Any Power of a Fraction.

We have, $\left(\frac{a}{b}\right)^3=\frac{a}{b}\times\frac{a}{b}\times\frac{a}{b}=\frac{a\times a\times a}{b\times b\times b}=\frac{a^3}{b^3}$;

and a similar result holds for any positive integral power of $\frac{a}{b}$.

Then, *a fraction may be raised to any power whose exponent is a positive integer by raising both numerator and denominator to the required power.*

Ex. $\left(-\frac{2x^4}{3y^3}\right)^5=-\left(\frac{2x^4}{3y^3}\right)^5$ (§ 55) $=-\frac{(2x^4)^5}{(3y^3)^5}=-\frac{32x^{20}}{243y^{15}}$ (§ 96).

EXERCISE 81

Find the values of the following:

1. $\left(\frac{6a^9b^6}{7c^8d^7}\right)^2$.
2. $\left(\frac{9mn^4}{8p^5}\right)^3$.
3. $\left(-\frac{3a^2x^7}{b^8y}\right)^5$.
4. $\left(-\frac{4x^m}{5y^4z^9}\right)^4$.
5. $\left(-\frac{2m^7x^2}{n^5y^4}\right)^7$.
6. $\left(\frac{a^3m^5}{3b^8cn^6}\right)^6$.

204. Square of a Polynomial.

We find by actual multiplication:

$$\begin{array}{llllll}
a & +b+c & & & & \\
a & +b+c & & & & \\
\hline
a^2+ & ab+ & ac & & & \\
 & +ab & & +b^2+ & bc & \\
 & & +ac & & +bc+ & c^2 \\
\hline
a^2+ & 2ab+ & 2ac+ & b^2+ & 2bc+ & c^2
\end{array}$$

The result, for convenience of enunciation, may be written:

$$(a+b+c)^2 = a^2+b^2+c^2+2\,ab+2\,ac+2\,bc.$$

In like manner we find:

$$(a+b+c+d)^2 = a^2+b^2+c^2+d^2$$
$$+2\,ab+2\,ac+2\,ad+2\,bc+2\,bd+2\,cd;$$

and so on.

We then have the following rule:

The square of a polynomial is equal to the sum of the squares of its terms, together with twice the product of each term by each of the following terms.

Ex. Expand $(2\,x^2-3\,x-5)^2$.

The squares of the terms are $4\,x^4$, $9\,x^2$, and 25.

Twice the product of the first term by each of the following terms gives the results $-12\,x^3$ and $-20\,x^2$.

Twice the product of the second term by the following term gives the result $30\,x$.

Then, $$(2\,x^2-3\,x-5)^2 = 4\,x^4+9\,x^2+25-12\,x^3-20\,x^2+30\,x$$
$$= 4\,x^4-12\,x^3-11\,x^2+30\,x+25.$$

EXERCISE 82

Square each of the following:

1. $a-b+c$.

2. $x+y-z$.

3. $n^2-3\,n-1$.

4. $3\,x+y+2\,z$.

5. $1+3\,x-4\,x^2$.

6. $2\,a^m-5\,a^n-1$.

7. $4+3\,m^3+2\,m^6$.

8. $7\,n^3-n^2+6$.

9. $2\,x+3\,y+5\,z$.

10. $x^2-4\,xy-5\,y^2$.

11. $6\,a^2+ab-3\,b^2$.

12. $2\,a^{3p}-8\,a+9$.

13. $6\,x^6-4\,x^3y^2+5\,y^4$.

14. $a+b-c-d$.

15. $a-b+c+d$.

16. a^3+a^2+a-3.

17. $2\,x^3-4\,x^2-3\,x+1$.

18. $3-2\,a+4\,a^2-5\,a^3$.

19. $m + 4 - \frac{2}{m}$.

20. $\frac{4a^3}{3} - \frac{a}{x} + \frac{2}{ax^3}$.

205. Cube of a Binomial.

We find by actual multiplication:

$$\begin{array}{rl} (a+b)^2 = & a^2 + 2ab + b^2 \\ & a + b \\ \hline & a^3 + 2a^2b + ab^2 \\ & \quad\quad a^2b + 2ab^2 + b^3 \\ \hline (a+b)^3 = & a^3 + 3a^2b + 3ab^2 + b^3 \end{array}$$

That is, *the cube of the sum of two numbers is equal to the cube of the first, plus three times the square of the first times the second, plus three times the first times the square of the second, plus the cube of the second.*

Again,

$$\begin{array}{rl} (a-b)^2 = & a^2 - 2ab + b^2 \\ & a - b \\ \hline & a^3 - 2a^2b + ab^2 \\ & \quad - a^2b + 2ab^2 - b^3 \\ \hline (a-b)^3 = & a^3 - 3a^2b + 3ab^2 - b^3 \end{array}$$

That is, *the cube of the difference of two numbers is equal to the cube of the first, minus three times the square of the first times the second, plus three times the first times the square of the second, minus the cube of the second.*

1. Find the cube of $a + 2b$

We have, $(a + 2b)^3 = a^3 + 3a^2(2b) + 3a(2b)^2 + (2b)^3$

$$= a^3 + 6a^2b + 12ab^2 + 8b^3.$$

2. Find the cube of $2x^3 - 5y^2$.

$$(2x^3 - 5y^2)^3 = (2x^3)^3 - 3(2x^3)^2(5y^2) + 3(2x^3)(5y^2)^2 - (5y^2)^3$$
$$= 8x^9 - 60x^6y^2 + 150x^3y^4 - 125y^6.$$

The cube of a *trinomial* may be found by the above method, if two of its terms be enclosed in parentheses, and regarded as a single term.

3. Find the cube of $x^2 - 2x - 1$.

$$(x^2 - 2x - 1)^3 = [(x^2 - 2x) - 1]^3$$
$$= (x^2 - 2x)^3 - 3(x^2 - 2x)^2 + 3(x^2 - 2x) - 1$$
$$= x^6 - 6x^5 + 12x^4 - 8x^3 - 3(x^4 - 4x^3 + 4x^2) + 3(x^2 - 2x) - 1$$
$$= x^6 - 6x^5 + 12x^4 - 8x^3 - 3x^4 + 12x^3 - 12x^2 + 3x^2 - 6x - 1$$
$$= x^6 - 6x^5 + 9x^4 + 4x^3 - 9x^2 - 6x - 1.$$

EXERCISE 83

Cube each of the following:

1. $a^2b - ab^2$.
2. $a + 3$.
3. $2x + y$.
4. $a - 5b$.
5. $6x^2 + 1$.
6. $m - 4n^3$.
7. $2a + 6x$.
8. $5m^p - 3n^q$.
9. $3a - 6a^2$.
10. $2a^4b + 7c^5$.
11. $8a^{2m} + 3a$.
12. $9x^7 - 4y^6$.
13. $\frac{m^2}{2n} - \frac{2n}{m^2}$.
14. $a - b + c$.
15. $1 - a - a^2$.
16. $a + 2b + c$.
17. $x^2 + x - 3$.
18. $3 - n + 2n^2$.
19. $2a^2 + 3a - 4$.

XVI. EVOLUTION

206. If an expression when raised to the nth power, n being a positive integer, is equal to another expression, the first expression is said to be the nth **Root** of the second.

Thus, if $a^n = b$, a is the nth root of b.

Evolution is the process of finding any required root of an expression.

207. The **Radical Sign**, $\sqrt{\ }$, when written before an expression, indicates some root of the expression.

Thus, $\sqrt{a}$ indicates the *second*, or *square* root of a;
$\sqrt[3]{a}$ indicates the *third*, or *cube* root of a;
$\sqrt[4]{a}$ indicates the *fourth* root of a; and so on.

The *index* of a root is the number written over the radical sign to indicate what root of the expression is taken.

If no index is expressed, the index 2 is understood.

An *even* root is one whose index is an even number; an *odd* root is one whose index is an odd number.

EVOLUTION OF MONOMIALS

208. We will now show how to find any root of a monomial, which is a perfect power of the same degree as the index of the required root.

1. Required the cube root of $a^3b^9c^6$.

We have, $(ab^3c^2)^3 = a^3b^9c^6$.

Then, by § 206, $\sqrt[3]{a^3b^9c^6} = ab^3c^2$.

2. Required the fifth root of $-32\,a^5$.

We have, $(-2\,a)^5 = -32\,a^5$.

Whence, $\sqrt[5]{-32\,a^5} = -2\,a$.

3. Required the fourth root of a^4.

We have either $(+a)^4$ or $(-a)^4$ equal to a^4.

Whence, $\sqrt[4]{a^4} = \pm a.$

The sign $\pm$, called the *double sign*, is prefixed to an expression when we wish to indicate that it is either $+$ or $-$.

209. From § 208, we have the following rule:

Extract the required root of the absolute value of the numerical coefficient, and divide the exponent of each letter by the index of the required root.

Give to every even root of a positive term the sign $\pm$, and to every odd root of any term the sign of the term itself.

1. Find the square root of $9a^4b^6c^{10}$.

By the rule, $\sqrt{9a^4b^6c^{10}} = \pm 3a^2b^3c^5.$

2. Find the cube root of $-64x^6y^{3n}$.

$$\sqrt[3]{-64x^6y^{3n}} = -4x^2y^n.$$

The root of a large number may sometimes be found by resolving it into its prime factors.

3. Find the square root of 254016.

We have, $\sqrt{254016} = \sqrt{2^6 \times 3^4 \times 7^2} = \pm 2^3 \times 3^2 \times 7 = \pm 504.$

4. Find the value of $\sqrt[3]{72 \times 75 \times 135}$.

$$\sqrt[3]{72 \times 75 \times 135} = \sqrt[3]{(2^3 \times 3^2) \times (3 \times 5^2) \times (3^3 \times 5)}$$
$$= \sqrt[3]{2^3 \times 3^6 \times 5^3} = 2 \times 3^2 \times 5 = 90.$$

EXERCISE 84

Find the values of the following:

1. $\sqrt{36x^2y^8}$.
2. $\sqrt[3]{64a^{12}b^3c^6}$.
3. $\sqrt[9]{-x^9y^{18}z^{27}}$.
4. $\sqrt[4]{81n^{20}x^{12}y^4}$.
5. $\sqrt{121a^{12}b^{22}c^4}$.
6. $\sqrt[3]{-216x^{24}y^9z^{15}}$.
7. $\sqrt[6]{64a^6n^{24}}$.
8. $\sqrt[5]{-243x^{30}y^{15}}$.
9. $\sqrt{169x^{6m}y^{4n-2}}$.

10. $\sqrt[7]{128\, m^{14}n^{21}}$.
11. $\sqrt[3]{343\, x^{3m+9}y^{6n}}$.
12. $\sqrt[4]{625\, a^{16m}b^{4n}}$.
13. $\sqrt{2916}$.
14. $\sqrt{30625}$.
15. $\sqrt{86436}$.
16. $\sqrt{81 \times 64 \times 324}$.
17. $\sqrt{84 \times 54 \times 126}$.
18. $\sqrt[3]{5832}$.
19. $\sqrt{15\, xy \times 33\, yz \times 55\, zx}$.
20. $\sqrt[3]{21952}$.
21. $\sqrt{627264}$.
22. $\sqrt[3]{112 \times 168 \times 252}$.
23. $\sqrt[4]{104976}$.
24. $\sqrt[5]{59049}$.
25. $\sqrt[4]{135 \times 375 \times 625}$.
26. $\sqrt{(a^2 - 5\, a + 6)(a^2 + 2\, a - 8)(a^2 + a - 12)}$.

210. Any Root of a Fraction.

It follows from § 203 that, to find any root of a fraction, each of whose terms is a perfect power of the same degree as the index of the required root, *extract the required root of both numerator and denominator.*

Ex.
$$\sqrt[3]{-\frac{27\, a^3b^6}{64\, c^9}} = -\frac{\sqrt[3]{27\, a^3b^6}}{\sqrt[3]{64\, c^9}} = -\frac{3\, ab^2}{4\, c^3}.$$

EXERCISE 85

Find the values of the following:

1. $\sqrt{\frac{64\, x^2y^8}{49\, z^4}}$.
2. $\sqrt[3]{-\frac{27\, a^9}{125\, b^6}}$.
3. $\sqrt[5]{\frac{32\, a^{25}}{b^5c^{20}}}$.
4. $\sqrt[7]{-\frac{128\, x^{14}}{y^{21}}}$.
5. $\sqrt[4]{\frac{m^{20}}{256\, n^8}}$.
6. $\sqrt[6]{\frac{a^{12m}}{729\, b^{6m}}}$.

211. We have $\sqrt[n]{(a^n)^m} = \sqrt[n]{a^{mn}} = a^m = (\sqrt[n]{a^n})^m$.

Ex. Required the value of $\sqrt[5]{(32\, a^{10})^4}$.

We have, $\sqrt[5]{(32\, a^{10})^4} = (\sqrt[5]{32\, a^{10}})^4 = (2\, a^2)^4 = 16\, a^8$.

This method of finding the root is shorter than raising $32\, a^{10}$ to the fourth power, and then taking the fifth root of the result.

EXERCISE 86

Find the values of the following:

1. $\sqrt[3]{(64\,a^3)^2}$.
2. $\sqrt{(4\,m^4)^7}$.
3. $\sqrt[4]{(16\,x^8y^{12})^5}$.
4. $\sqrt[5]{(-243\,a^5b^{25}c^{10})^3}$.
5. $\sqrt[6]{(64\,m^{12}n^6)^5}$.
6. $\sqrt[3]{\left(-\dfrac{8\,x^3}{27\,y^6}\right)^4}$.
7. $\sqrt{(a^2-2\,ab+b^2)^3}$.

SQUARE ROOT OF A POLYNOMIAL

212. In § 112, we showed how to find the square root of a trinomial perfect square.

The square roots of certain polynomials of the form

$$a^2+b^2+c^2+2\,ab+2\,ac+2\,bc$$

can be found by inspection.

Ex. Find the square root of

$$9\,x^2+y^2+4\,z^2+6\,xy-12\,xz-4\,yz.$$

We can write the expression as follows:

$$(3\,x)^2+y^2+(-2\,z)^2+2(3\,x)y+2(3\,x)(-2\,z)+2\,y(-2\,z).$$

By § 204, this is the square of $3\,x+y+(-2\,z)$.

Then, the square root of the expression is $3\,x+y-2\,z$.

(The result could also have been obtained in the form $2\,z-y-3\,x$.)

EXERCISE 87

Find the square roots of the following:

1. $a^2+b^2+c^2-2\,ab-2\,ac+2\,bc$.
2. $x^2+4\,y^2+9+4\,xy+6\,x+12\,y$.
3. $1+25\,m^2+36\,n^2-10\,m+12\,n-60\,mn$.
4. $a^2+81\,b^2+16+18\,ab-8\,a-72\,b$.
5. $9\,x^2+y^2+25\,z^2-6\,xy-30\,xz+10\,yz$.
6. $36\,m^2+64\,n^2+x^2+96\,mn-12\,mx-16\,nx$.
7. $16\,a^4+9\,b^4+81\,c^4+24\,a^2b^2+72\,a^2c^2+54\,b^2c^2$.
8. $25\,x^6+49\,y^{10}+36\,z^8-70\,x^3y^5+60\,x^3z^4-84\,y^5z^4$.

213. Square Root of any Polynomial Perfect Square.

By § 204, $(a+b+c)^2 = a^2 + 2ab + b^2 + 2ac + 2bc + c^2$

$$= a^2 + (2a+b)b + (2a+2b+c)c. \qquad (1)$$

Then, if the square of a trinomial be arranged in order of powers of some letter:

I. The square root of the first term gives the first term of the root, a.

II. If from (1) we subtract a^2, we have

$$(2a+b)b + (2a+2b+c)c. \qquad (2)$$

The first term of this, when expanded, is $2ab$; if this be divided by twice the first term of the root, $2a$, we have the next term of the root, b.

III. If from (2) we subtract $(2a+b)b$, we have

$$(2a+2b+c)c. \qquad (3)$$

The first term of this, when expanded, is $2ac$; if this be divided by twice the first term of the root, $2a$, we have the last term of the root, c.

IV. If from (3) we subtract $(2a+2b+c)c$, there is no remainder.

Similar considerations hold with respect to the square of a polynomial of any number of terms.

214. The principles of § 213 may be used to find the square root of a polynomial perfect square of any number of terms.

Let it be required to find the square root of

$$4x^4 + 12x^3 - 7x^2 - 24x + 16.$$

$$
\begin{array}{rl}
 & 4x^4 + 12x^3 - 7x^2 - 24x + 16 \;\underline{\big|\; 2x^2 + 3x - 4} \\
a^2 = & \underline{4x^4 \qquad\qquad\qquad\qquad\qquad} \\
2a + b = 4x^2 + 3x & \big|\; 12x^3 - 7x^2 - 24x + 16, \text{ 1st Rem.} \\
3x & \big|\; \underline{12x^3 + 9x^2 \qquad\qquad\quad} \\
2a + 2b + c = 4x^2 + 6x - 4 & \big|\; -16x^2 - 24x + 16, \text{ 2d Rem.} \\
-4 & \big|\; \underline{-16x^2 - 24x + 16}
\end{array}
$$

The first term of the root is the square root of $4x^4$, or $2x^2$.

Subtracting the square of $2x^2$, $4x^4$, from the given expression, the first remainder is $12x^3 - 7x^2 - 24x + 16$.

Dividing the first term of this by twice the first term of the root, $4x^2$, we have the next term of the root, $3x$ (§ 213, II).

Adding this to $4x^2$ gives $4x^2 + 3x$; multiplying the result by $3x$, and subtracting the product, $12x^3 + 9x^2$, from the first remainder, gives the second remainder, $-16x^2 - 24x + 16$.

Dividing the first term of this by twice the first term of the root, $4x^2$, we have the last term of the root, -4 (§ 213, III).

If from the second remainder we subtract $(4x^2 + 6x - 4)(-4)$, or $-16x^2 - 24x + 16$, there is no remainder; then, $2x^2 + 3x - 4$ is the required root (§ 213, IV).

The expressions $4x^2$ and $4x^2 + 6x$ are called ***trial-divisors***, and $4x^2 + 3x$ and $4x^2 + 6x - 4$ *complete divisors*.

We then have the following rule for extracting the square root of a polynomial perfect square:

Arrange the expression according to the powers of some letter.

Extract the square root of the first term, write the result as the first term of the root, and subtract its square from the given expression, arranging the remainder in the same order of powers as the given expression.

Divide the first term of the remainder by twice the first term of the root, and add the quotient to the part of the root already found, and also to the trial-divisor.

Multiply the complete divisor by the term of the root last obtained, and subtract the product from the remainder.

If other terms remain, proceed as before, doubling the part of the root already found for the next trial-divisor.

215. Examples.

1. Find the square root of $9x^4 + 30a^3x^2 + 25a^6$.

$$
\begin{array}{r|l}
 & 9x^4 + 30a^3x^2 + 25a^6 \;\underline{|\; 3x^2 + 5a^3} \\
 & 9x^4 \\ \hline
6x^2 + 5a^3 & 30a^3x^2 \\
 & \underline{30a^3x^2 + 25a^6}
\end{array}
$$

It is usual, in practice, to omit those terms, after the first, in each remainder, which are merely repetitions of the terms in the given expression; thus, in the first remainder of Ex. 1, we leave out the term $25a^6$

It is also usual to leave out of the written work the multiplier of the complete divisor.

2. Find the square root of

$$20x^2 - 22x^3 + 1 + 28x^4 + 9x^6 - 8x - 12x^5.$$

Arranging according to the descending powers of x, we have

$$\begin{array}{r|l}
 & 9x^6 - 12x^5 + 28x^4 - 22x^3 + 20x^2 - 8x + 1 \;\underline{\big|\, 3x^3 - 2x^2 + 4x - 1} \\
 & 9x^6 \\ \hline
6x^3 - 2x^2 & -12x^5 \\
 & -12x^5 + 4x^4 \\ \hline
6x^3 - 4x^2 + 4x & 24x^4 \\
 & 24x^4 - 16x^3 + 16x^2 \\ \hline
6x^3 - 4x^2 + 8x - 1 & -6x^3 + 4x^2 \\
 & -6x^3 + 4x^2 - 8x + 1 \\ \hline
\end{array}$$

It will be observed that *each trial-divisor is equal to the preceding complete divisor with its last term doubled.*

If, in Ex. 2, we had written the expression

$$1 - 8x + 20x^2 - 22x^3 + 28x^4 - 12x^5 + 9x^6,$$

the square root would have been obtained in the form $1 - 4x + 2x^2 - 3x^3$, which is the negative of $3x^3 - 2x^2 + 4x - 1$.

EXERCISE 88

Find the square roots of the following:

1. $x^4 + 6x^3 + 11x^2 + 6x + 1.$
2. $1 - 4a + 2a^2 + 4a^3 + a^4.$
3. $9n^4 + 12n^3 - 20n^2 - 16n + 16.$
4. $56x^2 + 4 - 60x^3 - 24x + 25x^4.$
5. $x^2 + y^2 + 4z^2 - 2xy + 4xz - 4yz.$
6. $8a^3 - 4a - 16a^4 + 1 + 16a^6 + 4a^2.$
7. $25x^6 + 10x^5y + x^4y^2 + 30x^3y^3 + 6x^2y^4 + 9y^6.$
8. $36a^2 + 25b^2 + 16c^2 - 60ab - 48ac + 40bc.$
9. $9x^4 + 6x^3y - 47x^2y^2 - 16xy^3 + 64y^4.$
10. $12n - 42n^3 + 4 - 19n^2 + 49n^4.$
11. $16a^4 + 48a^3b + 60a^2b^2 + 36ab^3 + 9b^4.$

12. $4n^{12} - 16n^9x^2 + 36n^6x^4 - 40n^3x^6 + 25x^8$.

13. $30x^6y^2 - 24x^2y^6 - 31x^4y^4 + 25x^8 + 16y^8$.

14. $4x^2 + 20x + 29 + \dfrac{10}{x} + \dfrac{1}{x^2}$.

15. $a^6 - 2a^5 - a^4 + 6a^3 - 3a^2 - 4a + 4$.

16. $5x^2 - 23x^4 + 12x + 8x^5 - 22x^3 + 16x^6 + 4$.

17. $a^2 - \dfrac{2ab}{3} + \dfrac{13b^2}{9} - \dfrac{4b^3}{9a} + \dfrac{4b^4}{9a^2}$.

18. $\dfrac{n^4}{4} - \dfrac{n^3}{3} - \dfrac{41n^2}{36} + \dfrac{5n}{6} + \dfrac{25}{16}$.

19. $9a^6 + 6a^5x + 31a^4x^2 - 14a^3x^3 + 17a^2x^4 - 40ax^5 + 16x^6$.

20. $\dfrac{x^4}{16} + \dfrac{x^3}{4y} + \dfrac{3x^2}{20y^2} - \dfrac{x}{5y^3} + \dfrac{1}{25y^4}$.

21. $\dfrac{25}{4} - \dfrac{15a}{b} + \dfrac{41a^2}{4b^2} - \dfrac{3a^3}{2b^3} + \dfrac{a^4}{16b^4}$.

22. $44a^3b^3 + 4b^6 - 30a^5b + 4a^2b^4 + 25a^6 - 16ab^5 - 31a^4b^2$.

SQUARE ROOT OF AN ARITHMETICAL NUMBER

216. The square root of 100 is 10; of 10000 is 100; etc.

Hence, the square root of a number between 1 and 100 is between 1 and 10; the square root of a number between 100 and 10000 is between 10 and 100; etc.

That is, the integral part of the square root of an integer of one or two digits, contains *one* digit; of an integer of three or four digits, contains *two* digits; and so on.

Hence, *if a point be placed over every second digit of an integer, beginning at the units' place, the number of points shows the number of digits in the integral part of its square root.*

217. Square Root of any Integral Perfect Square.

The square root of an integral perfect square may be found in the same way as the square root of a polynomial.

Required the square root of 106929.

$$
\begin{array}{rr|l}
 & 1\dot{0}6\dot{9}2\dot{9} & 300+20+7 \\
 & a^2 = 90000 & = a+b+c \\
\cline{2-2}
2a+b = & 600+20 \;\big|\; 16929 & \\
 & 20 \;\big|\; 12400 & \\
\cline{2-2}
2a+2b+c = & 600+40+7 \;\big|\; 4529 & \\
 & 7 \;\big|\; 4529 & \\
\end{array}
$$

Pointing the number in accordance with the rule of § 216, we find that there are three digits in its square root.

Let a represent the hundreds' digit of the root, with two ciphers annexed; b the tens' digit, with one cipher annexed; and c the units' digit.

Then, a must be the greatest multiple of 100 whose square is less than 106929; this we find to be 300.

Subtracting a^2, or 90000, from the given number, the result is 16929.

Dividing this remainder by $2a$, or 600, we have the quotient 28+; which suggests that b equals 20.

Adding this to $2a$, or 600, and multiplying the result by b, or 20, we have 12400; which, subtracted from 16929, leaves 4529.

Since this remainder equals $(2a+2b+c)c$ (§ 213, III), we can get c approximately by dividing it by $2a+2b$, or $600+40$.

Dividing 4529 by 640, we have the quotient 7+; which suggests that c equals 7.

Adding this to $600+40$, multiplying the result by 7, and subtracting the product, 4529, there is no remainder.

Then, $300+20+7$, or 327, is the required square root.

218. Omitting the ciphers for the sake of brevity, and condensing the operation, we may arrange the work of the example of § 217 as follows:

$$
\begin{array}{r|l}
 & 1\dot{0}6\dot{9}2\dot{9} \;\big|\; \underline{327} \\
 & 9 \\
\hline
62 & 169 \\
 & 124 \\
\hline
647 & 4529 \\
 & 4529 \\
\end{array}
$$

The numbers 600 and 640 are called *trial-divisors*, and the numbers 620 and 647 are called *complete divisors.*

We then have the following rule for finding the square root of an integral perfect square:

Separate the number into periods by pointing every second digit, beginning with the units' place.

Find the greatest square in the left-hand period, and write its square root as the first digit of the root; subtract the square of the first root-digit from the left-hand period, and to the result annex the next period.

Divide this remainder, omitting the last digit, by twice the part of the root already found, and annex the quotient to the root, and also to the trial-divisor.

Multiply the complete divisor by the root-digit last obtained, and subtract the product from the remainder.

If other periods remain, proceed as before, doubling the part of the root already found for the next trial-divisor.

Note 1. It sometimes happens that, on multiplying a complete divisor by the digit of the root last obtained, the product is greater than the remainder.

In such a case, the digit of the root last obtained is too great, and one less must be substituted for it.

Note 2. If any root-digit is 0, annex 0 to the trial-divisor, and annex to the remainder the next period. (See the illustrative example of § 220.)

219. *Ex.* Find the square root of 4624.

$$\begin{array}{r|l|l}
 & 4\dot{6}2\dot{4} & 68 \\
 & 36 & \\
\hline
128 & 1024 & \\
 & 1024 & \\
\hline
\end{array}$$

The greatest square in the left-hand period is 36.

Then the first digit of the root is 6.

Subtracting 6^2, or 36, from the left-hand period, the result is 10; to this we annex the next period, 24.

Dividing this remainder, omitting the last digit, or 102, by twice the part of the root already found, or 12, the quotient is 8; this we annex to the root, and also to the trial-divisor.

Multiplying the complete divisor, 128, by 8, and subtracting the product from the remainder, there is no remainder.

Then, 68 is the required square root.

220. We will now show how to find the square root of a number which is not integral.

Ex. Find the square root of 49.449024.

We have, $\sqrt{49.449024} = \sqrt{\dfrac{49449024}{1000000}} = \dfrac{\sqrt{49449024}}{\sqrt{1000000}}.$

$$\begin{array}{r|l}
 & 4\dot{9}4\dot{4}9\dot{0}2\dot{4} \;\underline{|\,7032} \\
 & 49 \\
\hline
1403 & 4490 \\
 & 4209 \\
\hline
14062 & 28124 \\
 & \underline{28124}
\end{array}$$

Since 14 is not contained in 4, we write 0 as the second root-digit, in the above example; we then annex 0 to the trial-divisor 14, and annex to the remainder the next period, 90. (See Note 2, § 218.)

Then, $\sqrt{49.449024} = \dfrac{7032}{1000} = 7.032.$

The work may be arranged as follows:

$$\begin{array}{r|l}
 & 4\dot{9}.4\dot{4}9\dot{0}2\dot{4} \;\underline{|\,7.032} \\
 & 49 \\
\hline
1403 & 4490 \\
 & 4209 \\
\hline
14062 & 28124 \\
 & \underline{28124}
\end{array}$$

Then, *if a point be placed over every second digit of any number, beginning with the units' place, and extending in either direction, the rule of* § 218 *may be applied to the result and the decimal point inserted in its proper position in the root.*

EXERCISE 89

Find the square roots of the following:

1. 5776. 2. 15376. 3. 67081.

4. 8427.24.	**8.** 7974.49.	**12.** .30316036.
5. .165649.	**9.** .00459684.	**13.** 39.375625.
6. .133225.	**10.** 22014864.	**14.** .000064272289.
7. 54.4644.	**11.** 1488.4164.	**15.** 889060.41.

221. Approximate Square Roots.

If there is a final remainder, the number has no exact square root; but we may continue the operation by annexing periods of ciphers, and obtain an approximate root, correct to any desired number of decimal places.

Ex. Find the square root of 12 to four decimal places.

$$\begin{array}{r|l}
 & 12.00000000 \;\underline{|\; 3.4641+} \\
 & 9 \\
\hline
64 & 3\,00 \\
 & 2\,56 \\
\hline
686 & 4400 \\
 & 4116 \\
\hline
6924 & 28400 \\
 & 27696 \\
\hline
69281 & 70400
\end{array}$$

222. The approximate square root of a fraction may be obtained by taking the square root of the numerator, and then of the denominator, and dividing the first result by the second.

If the denominator is not a perfect square, it is better to reduce the fraction to an equivalent fraction whose denominator is a perfect square.

Ex. Find the value of $\sqrt{\frac{3}{8}}$ to five decimal places.

We have, $\sqrt{\frac{3}{8}}=\sqrt{\frac{6}{16}}=\frac{\sqrt{6}}{\sqrt{16}}=\frac{2.44948+}{4}=.61237+.$

EXERCISE 90

Find the first five figures of the square root of:

1. 2.	**4.** 17.	**7.** .3.	**10.** .008.
2. 5.	**5.** 59.	**8.** .067.	**11.** .00095.
3. 11.	**6.** 75.8.	**9.** .46.	**12.** 96.756.

Find the first four figures of the square root of:

13. $\frac{7}{4}$.	15. $\frac{43}{49}$.	17. $\frac{4}{5}$.	19. $\frac{17}{32}$.	21. $\frac{23}{27}$.
14. $\frac{8}{9}$.	16. $\frac{1}{3}$.	18. $\frac{5}{8}$.	20. $\frac{16}{15}$.	22. $\frac{19}{48}$.

CUBE ROOT OF A POLYNOMIAL

223. The cube roots of certain polynomials of the form

$$a^3 + 3a^2b + 3ab^2 + b^3$$

can be found by inspection.

Ex. Find the cube root of $8a^3 - 36a^2b^2 + 54ab^4 - 27b^6$.

We can write the expression as follows:

$$(2a)^3 - 3(2a)^2(3b^2) + 3(2a)(3b^2)^2 - (3b^2)^3.$$

By § 205, this is the cube of $2a - 3b^2$.
Then, the cube root of the expression is $2a - 3b^2$.

EXERCISE 91

Find the cube roots of the following:

1. $a^3 + 6a^2 + 12a + 8$.
2. $1 - 9m + 27m^2 - 27m^3$.
3. $64n^3 - 48n^2 + 12n - 1$.
4. $125x^3 + 75x^2y + 15xy^2 + y^3$.
5. $a^6 + 18a^4b^3 + 108a^2b^6 + 216b^9$.
6. $125m^3 + 150m^2n + 60mn^2 + 8n^3$.
7. $27a^3b^3 - 108a^2b^2c + 144abc^2 - 64c^3$.
8. $m^6 - 21m^4x^4 + 147m^2x^8 - 343x^{12}$.

224. Cube Root of any Polynomial Perfect Cube.

By § 205, $(a + b + c)^3 = [(a + b) + c]^3$

$$= (a + b)^3 + 3(a + b)^2c + 3(a + b)c^2 + c^3$$

$$= a^3 + 3a^2b + 3ab^2 + b^3 + 3(a + b)^2c + 3(a + b)c^2 + c^3$$

$$= a^3 + (3a^2 + 3ab + b^2)b + [3(a + b)^2 + 3(a + b)c + c^2]c. \quad (1)$$

Then, if the cube of a trinomial be arranged in order of powers of some letter:

I. The cube root of the first term gives the first term of the cube root, a.

II. If from (1) we subtract a^3, we have

$$(3a^2+3ab+b^2)b+[3(a+b)^2+3(a+b)c+c^2]c. \qquad (2)$$

The first term of this, when expanded, is $3a^2b$; if this be divided by three times the square of the first term of the root, $3a^2$, we have the next term of the root, b.

III. If from (2) we subtract $(3a^2+3ab+b^2)b$, we have

$$[3(a+b)^2+3(a+b)c+c^2]c. \qquad (3)$$

The first term of this, when expanded, is $3a^2c$; if this be divided by three times the square of the first term of the root, $3a^2$, we have the last term of the root, c.

IV. If from (3) we subtract $[3(a+b)^2+3(a+b)c+c^2]c$, there is no remainder.

Similar considerations hold with respect to the cube of polynomial of any number of terms.

225. The principles of § 224 may be used to find the cube root of a polynomial perfect cube of any number of terms.

Let it be required to find the cube root of

$$x^6+6x^5+3x^4-28x^3-9x^2+54x-27.$$

$$\begin{array}{rr|l}
 & & \underline{x^2+2x-3} \\
 & & x^6+6x^5+3x^4-28x^3-9x^2+54x-27 \\
 & a^3 = x^6 & \\
\hline
3a^2+3ab+b^2= & 3x^4+6x^3+4x^2 & 6x^5+3x^4-28x^3-9x^2+54x-27 \\
 & 2x & 6x^5+12x^4+8x^3 \\
\hline
3(a+b)^2= & 3x^4+12x^3+12x^2 & -9x^4-36x^3-9x^2+54x-27 \\
3(a+b)c+c^2= & -9x^2-18x+9 & \\
\hline
 & 3x^4+12x^3+3x^2-18x+9 & \\
 & -3 & -9x^4-36x^3-9x^2+54x-27 \\
\hline
\end{array}$$

The first term of the root is the cube root of x^6, or x^2.

Subtracting the cube of x^2, or x^6, from the given expression, the first remainder is $6x^5+3x^4-28x^3-9x^2+54x-27$.

Dividing the first term of this by three times the square of the first term of the root, $3x^4$, we have the next term of the root, $2x$ (§ 224, II).

Now, $3ab+b^2$ equals $3\times x^2\times 2x+(2x)^2$, or $6x^3+4x^2$.

Adding this to $3x^4$, multiplying the result by $2x$, and subtracting the product, $6x^5+12x^4+8x^3$, from the first remainder, gives the second remainder, $-9x^4-36x^3-9x^2+54x-27$ (§ 224, III).

Dividing the first term of this by three times the square of the first term of the root, $3x^2$, we have the last term of the root, -3.

Now, $3(a+b)^2$ equals $3(x^2+2x)^2$, or $3x^4+12x^3+12x^2$; $3(a+b)c$ equals $3(x^2+2x)(-3)$, or $-9x^2-18x$; and $c^2=9$.

Adding these results, we have $3x^4+12x^3+3x^2-18x+9$.

Subtracting from the second remainder the product of this by -3, or $-9x^4-36x^3-9x^2+54x-27$, there is no remainder; then, x^2+2x-3 is the required root (§ 224, IV).

The expressions $3x^4$ and $3x^4+12x^3+12x^2$ are called *trial-divisors*, and the expressions $3x^4+6x^3+4x^2$ and $3x^4+12x^3+3x^2-18x+9$ *complete divisors.*

We then have the following rule for finding the cube root of a polynomial perfect cube:

Arrange the expression according to the powers of some letter.

Extract the cube root of the first term, write the result as the first term of the root, and subtract its cube from the given expression; arranging the remainder in the same order of powers as the given expression.

Divide the first term of the remainder by three times the square of the first term of the root, and write the result as the next term of the root.

Add to the trial-divisor three times the product of the term of the root last obtained by the part of the root previously found, and the square of the term of the root last obtained.

Multiply the complete divisor by the term of the root last obtained, and subtract the product from the remainder.

If other terms remain, proceed as before, taking three times the square of the part of the root already found for the next trial-divisor.

226. Examples.

1. Find the cube root of $8x^6-36x^4y+54x^2y^2-27y^3$.

$$
\begin{array}{r|l}
 & 8x^6 - 36x^4y + 54x^2y^2 - 27y^3 \;\underline{|\,2x^2 - 3y} \\
 & 8x^6 \\ \hline
12x^4 - 18x^2y + 9y^2 & -36x^4y \\
 & \underline{-36x^4y + 54x^2y^2 - 27y^3}
\end{array}
$$

It is usual, in practice, to omit those terms, after the first, in each remainder, which are merely repetitions of the terms in the given expression; and also to leave out of the written work the multiplier of the complete divisor.

2. Find the cube root of $40x^3 - 6x^5 - 64 + x^6 - 96x$.

Arranging according to the descending powers of x, we have

$$
\begin{array}{r|l}
 & x^6 - 6x^5 + 40x^3 - 96x - 64 \;\underline{|\,x^2 - 2x - 4} \\
 & x^6 \\ \hline
3x^4 - 6x^3 + 4x^2 & -6x^5 \\
 & -6x^5 + 12x^4 - 8x^3 \\ \hline
3x^4 - 12x^3 + 12x^2 & -12x^4 + 48x^3 \\
-12x^2 + 24x + 16 & \\ \hline
3x^4 - 12x^3 \qquad + 24x + 16 & \underline{-12x^4 + 48x^3 - 96x - 64}
\end{array}
$$

EXERCISE 92

Find the cube roots of the following:

1. $27x^6 + 27x^4 + 9x^2 + 1$. 2. $8a^6 - 60a^4b + 150a^2b^2 - 125b^3$.
3. $336xy^2 + 343x^3 - 64y^3 - 588x^2y$.
4. $x^6 + 9x^5 + 30x^4 + 45x^3 + 30x^2 + 9x + 1$.
5. $8n^6 - 12n^5 - 30n^4 + 35n^3 + 45n^2 - 27n - 27$.
6. $9a^4 + 54a^5 - 1 - 28a^3 - 3a^2 + 27a^6 + 6a$.
7. $\dfrac{a^3}{27} + \dfrac{a^2b}{12} + \dfrac{ab^2}{16} + \dfrac{b^3}{64}$.
8. $n^6 - 12n^5x + 57n^4x^2 - 136n^3x^3 + 171n^2x^4 - 108nx^5 + 27x^6$.
9. $135a^2b^4 + 12a^5b - 125b^6 + 8a^6 - 59a^3b^3 + 75ab^5 - 54a^4b^2$.
10. $152x^3 - 27 - 63x^2 + 27x^6 + 63x^4 - 108x - 108x^5$.
11. $\dfrac{a^3}{8} - \dfrac{15a^2}{4} + \dfrac{153a}{4} - 140 + \dfrac{153}{2a} - \dfrac{15}{a^2} + \dfrac{1}{a^3}$.
12. $64m^6 + 144m^5 + 204m^4 + 171m^3 + 102m^2 + 36m + 8$.
13. $x^9 - 6x^8 + 15x^7 - 26x^6 + 39x^5 - 42x^4 + 37x^3 - 30x^2 + 12x - 8$.

CUBE ROOT OF AN ARITHMETICAL NUMBER

227. The cube root of 1000 is 10; of 1000000 is 100; etc. Hence, the cube root of a number between 1 and 1000 is between 1 and 10; the cube root of a number between 1000 and 1000000 is between 10 and 100; etc.

That is, the integral part of the cube root of an integer of one, two, or three digits, contains *one* digit; of an integer of four, five, or six digits, contains *two* digits; and so on.

Hence, *if a point be placed over every third digit of an integer, beginning at the units' place, the number of points shows the number of digits in the integral part of its cube root.*

228. Cube Root of any Integral Perfect Cube.

The cube root of an integral perfect cube may be found in the same way as the cube root of a polynomial.

Required the cube root of 12487168.

$$
\begin{array}{rrl|l}
 & & 1\dot{2}48\dot{7}16\dot{8} & 200+30+2 \\
 & & a^3 = 8000000 & = a+b+c \\
\hline
3a^2 = & 120000 & 4487168 & \\
3ab = & 18000 & & \\
b^2 = & 900 & & \\
\hline
 & 138900 & & \\
 & 30 & 4167000 & \\
\hline
3(a+b)^2 = & 158700 & 320168 & \\
3(a+b)c = & 1380 & & \\
c^2 = & 4 & & \\
\hline
 & 160084 & & \\
 & 2 & 320168 & \\
\hline
\end{array}
$$

Pointing the number in accordance with the rule of § 227, we find that there are three digits in the cube root.

Let a represent the hundreds' digit of the root, with two ciphers annexed; b the tens' digit, with one cipher annexed; and c the units' digit.

Then, a must be the greatest multiple of 100 whose cube is less than 12487168; this we find to be 200.

Subtracting a^3, or 8000000, from the given number, the result is 4487168.

Dividing this by $3\,a^2$, or 120000, we have the quotient 37+; which suggests that b equals 30.

Adding to the divisor 120000, $3\,ab$, or 18000, and b^2, or 900, we have 138900.

Multiplying this by b, or 30, and subtracting the product 4167000 from 4487168, we have 320168.

Since this remainder equals $[3(a+b)^2+3(a+b)c+c^2]c$ (§ 224, III), we can get c approximately by dividing it by $3(a+b)^2$, or 158700.

Dividing 320168 by 158700, the quotient is 2+; which suggests that c equals 2.

Adding to the divisor 158700, $3(a+b)c$, or 1380, and c^2, or 4, we have 160084; multiplying this by 2, and subtracting the product, 320168, there is no remainder.

Then, $200+30+2$, or 232, is the required cube root.

229. Omitting the ciphers for the sake of brevity, and condensing the process, the work of the example of § 228 will stand as follows:

$$
\begin{array}{r|l}
\multicolumn{2}{r}{1\dot{2}48\dot{7}16\dot{8}\ \underline{|\,232}} \\
\multicolumn{2}{r}{8} \\
\hline
1200 & 4487 \\
180 & \\
9 & \\
\hline
1389 & 4167 \\
\hline
158700 & 320168 \\
1380 & \\
4 & \\
\hline
160084 & 320168 \\
\hline
\end{array}
$$

The numbers 120000 and 158700 are called *trial-divisors*, and the numbers 138900 and 160084 are called *complete divisors*.

We then have the following rule for finding the cube root of an integral perfect cube:

Separate the number into periods by pointing every third digit, beginning with the units' place.

Find the greatest cube in the left-hand period, and write its cube root as the first digit of the root; subtract the cube of the first root-digit from the left-hand period, and to the result annex the next period.

Divide this remainder by three times the square of the part of the root already found, with two ciphers annexed, and write the quotient as the next digit of the root.

Add to the trial-divisor three times the product of the last root-digit by the part of the root previously found, with one cipher annexed, and the square of the last root-digit.

Multiply the complete divisor by the digit of the root last obtained, and subtract the product from the remainder.

If other periods remain, proceed as before, taking three times the square of the part of the root already found, with two ciphers annexed, for the next trial-divisor.

Note 1. Note 1, § 218, applies with equal force to the above rule.

Note 2. If any root-figure is 0, annex two ciphers to the trial-divisor, and annex to the remainder the next period.

230. In the example of § 228, the first complete divisor is

$$3a^2 + 3ab + b^2. \qquad (1)$$

The next trial-divisor is $3(a+b)^2$, or $3a^2 + 6ab + 3b^2$.

This may be obtained from (1) by adding to it its second term, and double its third term.

That is, *if the first number and the double of the second number required to complete any trial-divisor be added to the complete divisor, the result, with two ciphers annexed, will give the next trial-divisor.*

This rule saves much labor in forming the trial-divisors.

231. *Ex.* Find the cube root of 157464.

	$15\dot{7}46\dot{4}$	54
	125	
7500	32464	
600		
16		
8116	32464	

232. We will now show how to find the cube root of a number which is not integral.

Ex. Find the cube root of 8144.865728.

We have, $\sqrt[3]{8144.865728}=\sqrt[3]{\dfrac{8144865728}{1000000}}=\dfrac{\sqrt[3]{8144865728}}{\sqrt[3]{1000000}}.$

$$
\begin{array}{r|l}
 & \dot{8}14\dot{4}86\dot{5}72\dot{8} \;\underline{|\,2012} \\
 & 8 \\ \hline
120000 & 144865 \\
600 & \\
\underline{1} & \\
120601 & \underline{120601} \\
600 & 24264728 \\
\underline{2} & \\
12120300 & \\
12060 & \\
\underline{4} & \\
12132364 & \underline{24264728}
\end{array}
$$

Since 1200 is not contained in 144, we write 0 as the second root-digit, in the above example; we then annex two ciphers to the trial-divisor 1200, and annex to the remainder the next period, 865. (Note 1, § 229.)

The second trial-divisor is formed by the rule of § 230.

Adding to the complete divisor 120601 the first number, 600, and twice the second number, 2, required to complete the trial-divisor 120000, we have 121203; annexing two ciphers to this, the result is 12120300.

Then, $\sqrt[3]{8144.865728}=\dfrac{2012}{100}=20.12.$

The work may be arranged as follows:

$$
\begin{array}{r|l}
 & \dot{8}14\dot{4}.86\dot{5}72\dot{8} \;\underline{|\,20.12} \\
 & 8 \\ \hline
120000 & 144\,865 \\
600 & \\
\underline{1} & \\
120601 & \underline{120\,601} \\
600 & 24\,264728 \\
\underline{2} & \\
12120300 & \\
12060 & \\
\underline{4} & \\
12132364 & \underline{24\,264728}
\end{array}
$$

It follows from the above that, *if a point be placed over every third digit of any number, beginning with the units' place, and extending in either direction, the rule of § 229 may be applied to the result, and the decimal point inserted in its proper position in the root.*

EXERCISE 93

Find the cube roots of the following:

1. 54872.	**6.** 3176523.	**11.** 331373888.
2. 262144.	**7.** 130323.843.	**12.** 37.595375.
3. 103.823.	**8.** 102503232.	**13.** 667627.624.
4. .884736.	**9.** 000356400829.	**14.** .964430272.
5. .000493039.	**10.** 22.665187.	**15.** 3422470.843.

Find the first four figures of the cube root of:

16. 4.	**18.** 8.2.	**20.** $\frac{7}{8}$.	**22.** $\frac{5}{16}$.
17. 9.	**19.** .03.	**21.** $\frac{13}{27}$.	**23.** $\frac{2}{7}$.

233. If the index of the required root is the product of two or more numbers, we may obtain the result by *successive extractions of the simpler roots.*

For by § 206, $(\sqrt[mn]{a})^{mn} = a.$

Taking the nth root of both members,

$$(\sqrt[mn]{a})^{m} = \sqrt[n]{a}. \qquad (1)$$

Taking the mth root of both members of (1),

$$\sqrt[mn]{a} = \sqrt[m]{\sqrt[n]{a}}.$$

Hence, the *mnth root of an expression is equal to the mth root of the nth root of the expression.*

Thus, to find the fourth root of an expression, we find the square root of its square root; to find the sixth root, we find the cube root of the square root, etc.

EXERCISE 94

Find the fourth roots of the following:

1. $a^8 - 16\,a^6b^3 + 96\,a^4b^6 - 256\,a^2b^9 + 256\,b^{12}$.

2. $81\,a^8 - 108\,a^7 + 162\,a^6 - 120\,a^5 + 91\,a^4 - 40\,a^3 + 18\,a^2 - 4\,a + 1$.

3. $16 + 32\,x - 72\,x^2 - 136\,x^3 + 145\,x^4 + 204\,x^5 - 162\,x^6 - 108\,x^7 + 81\,x^8$.

4. .011156640625.

Find the sixth roots of the following:

5. $64\,x^{12} + 192\,x^{10} + 240\,x^8 + 160\,x^6 + 60\,x^4 + 12\,x^2 + 1$.

6. $a^6 - 18\,a^5 + 135\,a^4 - 540\,a^3 + 1215\,a^2 - 1458\,a + 729$.

7. 34296.447249.

234. By § 206, $(\sqrt[n]{ab})^n = ab$.

Also, $(\sqrt[n]{a} \times \sqrt[n]{b})^n = (\sqrt[n]{a})^n \times (\sqrt[n]{b})^n = ab$.

Then, $(\sqrt[n]{ab})^n = (\sqrt[n]{a} \times \sqrt[n]{b})^n$.

Whence, $\sqrt[n]{ab} = \sqrt[n]{a} \times \sqrt[n]{b}$.

XVII. THEORY OF EXPONENTS

235. In the preceding portions of the work, an exponent has been considered only as a *positive integer.*

Thus, if m is a positive integer,

$$a^m = a \times a \times a \times \cdots \text{to } m \text{ factors.} \qquad (\S\ 11)$$

The following results have been proved to hold for any positive integral values of m and n:

$$a^m \times a^n = a^{m+n} \ (\S\ 56). \qquad (1)$$

$$(a^m)^n = a^{mn} \ (\S\ 93). \qquad (2)$$

236. It is necessary to employ exponents which are not positive integers; and we now proceed to define them, and prove the rules for their use.

In determining what meanings to assign to the new forms, it will be convenient to have them such that the above law for multiplication shall hold with respect to them.

We shall therefore assume equation (1), § 235, to hold for *all* values of m and n, and find what meanings must be attached in consequence to *fractional*, *negative*, and *zero* exponents.

237. Meaning of a Fractional Exponent.

Let it be required to find the meaning of $a^{\frac{5}{3}}$.

If (1), § 235, is to hold for all values of m and n,

$$a^{\frac{5}{3}} \times a^{\frac{5}{3}} \times a^{\frac{5}{3}} = a^{\frac{5}{3}+\frac{5}{3}+\frac{5}{3}} = a^5.$$

Then, the *third power* of $a^{\frac{5}{3}}$ equals a^5.

Hence, $a^{\frac{5}{3}}$ must be the *cube root* of a^5, or $a^{\frac{5}{3}} = \sqrt[3]{a^5}$.

We will now consider the general case.

Let it be required to find the meaning of $a^{\frac{p}{q}}$, where p and q are any positive integers.

If (1), § 235, is to hold for all values of m and n,

$$a^{\frac{p}{q}} \times a^{\frac{p}{q}} \times a^{\frac{p}{q}} \times \cdots \text{ to } q \text{ factors} = a^{\frac{p}{q}+\frac{p}{q}+\frac{p}{q}+\cdots \text{ to } q \text{ terms}} = a^{\frac{p}{q}\times q} = a^p.$$

Then, the qth power of $a^{\frac{p}{q}}$ equals a^p.

Hence, $a^{\frac{p}{q}}$ must be the qth root of a^p, or $a^{\frac{p}{q}} = \sqrt[q]{a^p}$.

Hence, *in a fractional exponent, the numerator denotes a power, and the denominator a root.*

For example, $a^{\frac{3}{4}} = \sqrt[4]{a^3}$; $b^{\frac{5}{2}} = \sqrt{b^5}$; $x^{\frac{1}{3}} = \sqrt[3]{x}$; etc.

EXERCISE 95

Express the following with radical signs:

1. $a^{\frac{5}{6}}$.
2. $x^{\frac{7}{3}}$.
3. $7m^{\frac{1}{4}}$.
4. $5x^{\frac{3}{5}}$.
5. $a^{\frac{2}{3}}b^{\frac{3}{5}}$.
6. $x^{\frac{6}{7}}y^{\frac{10}{9}}$.
7. $8a^{\frac{4}{5}}m^{\frac{3}{8}}$.
8. $10n^{\frac{5}{3}}x^{\frac{12}{7}}$.
9. $x^{\frac{3}{4}}y^{\frac{1}{6}}z^{\frac{11}{6}}$.
10. $2a^{\frac{1}{m}}b^{\frac{n}{2}}c^{\frac{3p}{q}}$.

Express the following with fractional exponents:

11. $\sqrt[7]{x^6}$.
12. $\sqrt[6]{a}$.
13. $\sqrt{m^3}$.
14. $\sqrt[3]{n^4}$.
15. $3\sqrt[4]{b^9}$.
16. $4\sqrt[9]{y^2}$.
17. $9\sqrt[5]{m}\sqrt[8]{n^5}$.
18. $\sqrt[6]{x^7}\sqrt[9]{y^8}$.
19. $\sqrt[8]{a}\sqrt[7]{b^{10}}$.
20. $\sqrt[3]{x^m}\sqrt{y^{3n}}\sqrt[2p]{z^q}$.

238. Meaning of a Zero Exponent.

If (1), § 235, is to hold for all values of m and n, we have

$$a^m \times a^0 = a^{m+0} = a^m.$$

Whence,
$$a^0 = \frac{a^m}{a^m} = 1.$$

We must then define a^0 as being equal to 1.

239. Meaning of a Negative Exponent.

Let it be required to find the meaning of a^{-3}.

If (1), § 235, is to hold for all values of m and n,

$$a^{-3} \times a^3 = a^{-3+3} = a^0 = 1 \text{ (§ 238)}.$$

Whence, $$a^{-3}=\frac{1}{a^3}.$$

We will now consider the general case.

Let it be required to find the meaning of a^{-s}, where s represents a positive integer or a positive fraction.

If (1), § 235, is to hold for all values of m and n,

$$a^{-s}\times a^{s}=a^{-s+s}=a^0=1 \text{ (§ 238)}.$$

Whence, $$a^{-s}=\frac{1}{a^s}.$$

We must then define a^{-s} as being equal to 1 divided by a^s.

For example, $a^{-2}=\frac{1}{a^2}$; $a^{-\frac{2}{3}}=\frac{1}{a^{\frac{2}{3}}}$; $3x^{-1}y^{-\frac{1}{2}}=\frac{3}{xy^{\frac{1}{2}}}$; etc.

240. It follows from § 239 that

Any factor of the numerator of a fraction may be transferred to the denominator, or any factor of the denominator to the numerator, if the sign of its exponent be changed.

Thus, $$\frac{a^2b^3}{cd^4}=\frac{b^3}{a^{-2}cd^4}=\frac{a^2b^3c^{-1}}{d^4}=\frac{a^2d^{-4}}{b^{-3}c}, \text{ etc.}$$

EXERCISE 96

Express with positive exponents:

1. $x^{-4}y^3$.
2. $a^{\frac{5}{2}}b^{-3}$.
3. $m^{-\frac{1}{4}}n^{\frac{1}{3}}$.
4. $3n^{-1}x$.
5. $a^{-2}m^{-3}$.
6. $m^{-\frac{3}{10}}x^9$.
7. $4a^{-\frac{4}{3}}n^{-5}$.
8. $5x^4y^{-\frac{3}{5}}z^{-7}$.
9. $m^{-\frac{2}{7}}n^{-9}$.
10. $8a^{-\frac{5}{4}}b^{-10}c^7$.
11. $6m^{-\frac{2}{3}}n^{-\frac{7}{6}}x^{\frac{3}{4}}$.
12. $7a^{-\frac{5}{2}}n^{-6}x^{-\frac{6}{5}}$.

Transfer all literal factors from the denominators to the numerators in the following:

13. $\frac{1}{x^3}$.
14. $\frac{a^{\frac{4}{3}}}{b^{-\frac{7}{3}}}$.
15. $\frac{1}{8m^{-2}n^{\frac{3}{2}}}$.
16. $\frac{3}{ax^{-4}}$.

17. $\dfrac{z^{-\frac{2}{5}}}{x^{-\frac{2}{7}}y^3}$. 18. $\dfrac{2\,m^5}{5\,np^{-1}}$. 19. $\dfrac{7\,a^{\frac{1}{3}}b^{-3}}{6\,c^{\frac{1}{3}}d^{\frac{5}{6}}}$. 20. $\dfrac{9\,m^{-7}x^{-\frac{5}{4}}}{4\,n^{-\frac{5}{3}}y^{-9}}$.

Transfer all literal factors from the numerators to the denominators in the following:

21. $\dfrac{4\,a^7}{3}$. 23. $\dfrac{m^{-2}n^{-\frac{1}{6}}}{2}$. 25. $\dfrac{6\,a^{-\frac{2}{3}}b^8}{c^6}$. 27. $\dfrac{9\,m^{-6}n^{-\frac{7}{8}}}{x^9y^{-\frac{1}{3}}}$.

22. $\dfrac{x^{\frac{4}{5}}}{y^{\frac{3}{7}}}$. 24. $\dfrac{xy^{-5}}{5}$. 26. $\dfrac{a^8m^{\frac{3}{7}}}{n^{-7}}$. 28. $\dfrac{8\,a^{-1}b^{\frac{5}{6}}}{5\,c^{-8}d^{-\frac{7}{8}}}$.

241. We obtained the definitions of fractional, zero, and negative exponents by supposing equation (1), § 235, to hold for such exponents.

Then, for any values of m and n,

$$a^m \times a^n = a^{m+n}. \qquad (1)$$

The formal proof of this result for positive or negative, integral or fractional, values of m and n will be found in § 445.

1. Find the value of $a^2 \times a^{-5}$.

We have, $a^2 \times a^{-5} = a^{2-5} = a^{-3}$.

2. Find the value of $a \times \sqrt{a^5}$.

By § 237, $a \times \sqrt{a^5} = a \times a^{\frac{5}{2}} = a^{1+\frac{5}{2}} = a^{\frac{7}{2}}$.

3. Multiply $a + 2\,a^{\frac{2}{3}} - 3\,a^{\frac{1}{3}}$ by $2 - 4\,a^{-\frac{1}{3}} - 6\,a^{-\frac{2}{3}}$.

$$\begin{array}{lllll}
a & +\,2\,a^{\frac{2}{3}} & -\,3\,a^{\frac{1}{3}} & & \\
2 & -\,4\,a^{-\frac{1}{3}} & -\,6\,a^{-\frac{2}{3}} & & \\
\hline
2\,a & +\,4\,a^{\frac{2}{3}} & -\,6\,a^{\frac{1}{3}} & & \\
 & -\,4\,a^{\frac{2}{3}} & -\,8\,a^{\frac{1}{3}} & +\,12 & \\
 & & -\,6\,a^{\frac{1}{3}} & -\,12 & +\,18\,a^{-\frac{1}{3}} \\
\hline
2\,a & & -\,20\,a^{\frac{1}{3}} & & +\,18\,a^{-\frac{1}{3}}
\end{array}$$

It must be carefully observed, in examples like the above, that the zero power of any number equals 1 (§ 238).

EXERCISE 97

Multiply the following:

1. a^3 by $a^{\frac{1}{3}}$.
2. x^{-6} by x^{-7}.
3. a^5 by a^{-5}.
4. n^3 by n^{-3}.
5. $3\,a^{-4}$ by $a^{-\frac{5}{3}}$.
6. m by $4\,m^{-\frac{13}{9}}$.
7. $2\,x^{-\frac{2}{3}}$ by $7\,x^{\frac{5}{3}}$.
8. x^{-2} by $\sqrt[4]{x^9}$.
9. $\sqrt[6]{a}$ by $a^{\frac{3}{4}}$.
10. $m^{\frac{1}{2}}$ by $\dfrac{5}{m^{\frac{5}{7}}}$.
11. $8\sqrt{a^{-3}}$ by $\sqrt[8]{a^{-5}}$.
12. $6\,a^2b$ by $a^{-9}b^{-\frac{3}{5}}$.
13. $3\,x^{-\frac{2}{3}}y^{-1}$ by $4\,x^{-\frac{1}{6}}y$.
14. $a^{-4}\sqrt[5]{x^2}$ by $a^{-3}\sqrt{x^5}$.
15. $m^{-1}n^{-\frac{3}{10}}$ by $\dfrac{1}{3\,m^{-\frac{9}{7}}n^{\frac{1}{5}}}$.
16. $x^{\frac{2}{3}}-2\,x^{\frac{1}{3}}y^{\frac{1}{3}}+4\,y^{\frac{2}{3}}$ by $x^{\frac{1}{3}}+2\,y^{\frac{1}{3}}$.
17. $2\,n^{-\frac{3}{4}}-5-6\,n^{\frac{3}{4}}$ by $3\,n^{-\frac{3}{4}}-4$.
18. $4\,a^{-4}+10\,a^{-2}+25$ by $2\,a^{-2}-5$.
19. $a^{\frac{1}{2}}-a^{\frac{1}{4}}b^{\frac{1}{4}}+b^{\frac{1}{2}}$ by $a^{\frac{1}{2}}+a^{\frac{1}{4}}b^{\frac{1}{4}}+b^{\frac{1}{2}}$.
20. $x^{-\frac{5}{6}}-x^{-\frac{2}{3}}y^{\frac{1}{3}}+2\,x^{-\frac{1}{2}}y^{\frac{2}{3}}$ by $x^{-\frac{1}{3}}y^{\frac{2}{3}}-x^{-\frac{1}{6}}y^{\frac{4}{3}}+2\,y$.
21. $a^{-\frac{5}{4}}b^3+4\,a^{-1}+3\,a^{-\frac{3}{4}}b^{-3}$ by $a^{-\frac{1}{2}}-4\,a^{-\frac{1}{4}}b^{-3}-3\,b^{-6}$.
22. $x^{\frac{3}{2}}-4\,x^{\frac{3}{4}}-5+6\,x^{-\frac{3}{4}}$ by $2\,x^{-\frac{3}{4}}+x^{-\frac{3}{2}}-3\,x^{-\frac{9}{4}}$.
23. $a^{-\frac{4}{3}}-2\,a^{-1}n^{-1}+3\,a^{-\frac{2}{3}}n^{-2}$ by $2\,a^{-\frac{2}{3}}n^{-1}+4\,a^{-\frac{1}{3}}n^{-2}-6\,n^{-3}$.
24. $2\,a^{\frac{5}{2}}x^{\frac{5}{6}}-a^2x-5\,a^{\frac{3}{2}}x^{\frac{7}{6}}$ by $4\,a^{-\frac{3}{2}}x^{-\frac{1}{3}}+2\,a^{-2}x^{-\frac{1}{6}}+10\,a^{-\frac{5}{2}}$.

242. *To prove* $\dfrac{a^m}{a^n}=a^{m-n}$ *for all values of m and n.*

By § 240, $\dfrac{a^m}{a^n}=a^m\times a^{-n}=a^{m-n}$, by (1), § 241.

The proof of this result in the case where m and n are positive integers, and $m>n$, is given in § 70.

1. Find the value of $\dfrac{a^{\frac{1}{2}}}{a^{-2}}$.

We have, $$\frac{a^{\frac{1}{2}}}{a^{-2}}=a^{\frac{1}{2}+2}=a^{\frac{5}{2}}.$$

2. Find the value of $\dfrac{a^{-3}}{\sqrt[5]{a^2}}$.

$$\frac{a^{-3}}{\sqrt[5]{a^2}}=\frac{a^{-3}}{a^{\frac{2}{5}}}=a^{-3-\frac{2}{5}}=a^{-\frac{17}{5}}.$$

3. Divide $18\,xy^{-2}-23+x^{-\frac{1}{2}}y+6\,x^{-1}y^2$ by $3\,x^{\frac{3}{4}}y^{-1}+x^{\frac{1}{4}}-2\,x^{-\frac{1}{4}}y$.

$$\begin{array}{l|l}
18\,xy^{-2}-23+x^{-\frac{1}{2}}y+6\,x^{-1}y^2 & 3\,x^{\frac{3}{4}}y^{-1}+x^{\frac{1}{4}}-2\,x^{-\frac{1}{4}}y \\ \cline{2-2}
\underline{18\,xy^{-2}+6\,x^{\frac{1}{2}}y^{-1}-12} & 6\,x^{\frac{1}{4}}y^{-1}-2\,x^{-\frac{1}{4}}-3\,x^{-\frac{3}{4}}y \\
\quad -6\,x^{\frac{1}{2}}y^{-1}-11+\ \ x^{-\frac{1}{2}}y+6\,x^{-1}y^2 & \\
\quad \underline{-6\,x^{\frac{1}{2}}y^{-1}-\ \ 2+4\,x^{-\frac{1}{2}}y\qquad\qquad} & \\
\qquad\qquad -\ \ 9-3\,x^{-\frac{1}{2}}y+6\,x^{-1}y^2 & \\
\qquad\qquad \underline{-\ \ 9-3\,x^{-\frac{1}{2}}y+6\,x^{-1}y^2} &
\end{array}$$

It is important to arrange the dividend, divisor, and each remainder in the same order of powers of some common letter.

EXERCISE 98

Divide the following:

1. x^5 by x^9.
2. $a^{\frac{4}{9}}$ by $a^{\frac{1}{3}}$.
3. n by $n^{-\frac{5}{6}}$.
4. $m^{-\frac{3}{4}}$ by $\sqrt[6]{m}$.
5. a^{-3} by $\dfrac{1}{\sqrt[3]{a^7}}$.
6. $x^{\frac{1}{5}}$ by $x^{-\frac{3}{10}}$.
7. $\sqrt{a^5}$ by $\sqrt[4]{a^{-3}}$.
8. $8\sqrt[4]{m^{-5}}$ by $2\,m^{-2}$.
9. $9a^{-4}b^{-\frac{2}{7}}$ by $3a^7b^{-\frac{4}{3}}$.
10. $x^{-\frac{3}{2}}-2\,x^{-\frac{1}{2}}-8\,x^{\frac{1}{2}}$ by $x^{-\frac{3}{4}}-4\,x^{\frac{1}{4}}$.
11. $a^{-1}-b^{-1}$ by $a^{-\frac{1}{3}}-b^{-\frac{1}{3}}$.
12. x^2-1 by $x^{\frac{1}{2}}+1$.
13. n^6-7+n^{-6} by n^3+3+n^{-3}.
14. $a^{-\frac{4}{3}}+4\,a^{-1}-2\,a^{-\frac{2}{3}}-12\,a^{-\frac{1}{3}}+9$ by $a^{-\frac{2}{3}}+2\,a^{-\frac{1}{3}}-3$.
15. $8\,m^{\frac{9}{2}}+12\,m^3n^{\frac{2}{3}}+6\,m^{\frac{3}{2}}n^{\frac{4}{3}}+n^2$ by $2\,m^{\frac{3}{2}}+n^{\frac{2}{3}}$.
16. $x^8y^{-12}-11\,x^4y^{-6}+1$ by $x^7y^{-8}+3\,x^5y^{-5}-x^3y^{-2}$.
17. $a^{-\frac{4}{3}}+2\,a^{-\frac{2}{3}}b^{-2}+9\,b^{-4}$ by $a^{-1}+2\,a^{-\frac{2}{3}}b^{-1}+3\,a^{-\frac{1}{3}}b^{-2}$.

18. $4\,a^{\frac{5}{4}}x^{-2} - 17\,a^{\frac{1}{4}}x^{2} + 16\,a^{-\frac{3}{4}}x^{6}$ by $2\,a^{\frac{1}{2}} - x^{2} - 4\,a^{-\frac{1}{2}}x^{4}$.

19. $9\,m^{\frac{7}{4}}n^{-\frac{1}{3}} - 10\,m^{\frac{3}{4}}n^{\frac{2}{3}} + m^{-\frac{1}{4}}n$ by $3\,m^{\frac{3}{2}}n^{\frac{1}{3}} - 4\,mn^{\frac{2}{3}} + m^{\frac{1}{2}}n$.

243. We will now show how to prove equation (2), § 235, for any values of m and n.

We will consider three cases, in each of which m may have any value, positive or negative, integral or fractional.

I. Let n be a positive integer.

The proof given in § 93 holds if n is a positive integer, whatever the value of m.

II. Let $n = \dfrac{p}{q}$, where p and q are positive integers.

Then, by the definition of § 237,

$$(a^m)^{\frac{p}{q}} = \sqrt[q]{(a^m)^p} = \sqrt[q]{a^{mp}} \text{ (§ 243, I)} = a^{\frac{mp}{q}}.$$

III. Let $n = -s$, where s is a positive number.

Then, by the definition of § 239,

$$(a^m)^{-s} = \frac{1}{(a^m)^s} = \frac{1}{a^{ms}} \text{ (§ 243, I or II)} = a^{-ms}.$$

Therefore, the result holds for all values of m and n.

1. Find the value of $(a^2)^{-5}$.

We have, $\qquad (a^2)^{-5} = a^{2\times -5} = a^{-10}$.

2. Find the value of $(a^{-3})^{-\frac{1}{3}}$.

$$(a^{-3})^{-\frac{1}{3}} = a^{-3\times -\frac{1}{3}} = a.$$

3. Find the value of $(\sqrt{a})^{\frac{2}{3}}$.

$$(\sqrt{a})^{\frac{2}{3}} = (a^{\frac{1}{2}})^{\frac{2}{3}} = a^{\frac{1}{2}\times\frac{2}{3}} = a^{\frac{1}{3}}.$$

EXERCISE 99

Find the values of the following:

1. $(a^3)^{-4}$. 2. $(x^{-6})^5$. 3. $(x^{\frac{3}{2}})^{\frac{2}{3}}$. 4. $(a^{-\frac{6}{7}})^{-\frac{1}{6}}$.

5. $(m^{-\frac{5}{4}})^8$. 8. $(x^2)^{-\frac{9}{10}}$. 11. $(\sqrt[6]{a^{-1}})^{-4}$. 14. $\left(\frac{1}{\sqrt[4]{m^7}}\right)^2$.

6. $(n^{\frac{3}{5}})^{-1}$. 9. $(\sqrt[5]{m^8})^{\frac{3}{4}}$. 12. $(\sqrt[9]{x^4})^{\frac{3}{8}}$. 15. $[(\sqrt[3]{x^2})^{\frac{1}{5}}]^{-\frac{3}{4}}$.

7. $(a^{-5})^{-\frac{7}{5}}$. 10. $\left(\frac{1}{n^2}\right)^7$. 13. $(x^{\frac{5m}{6}})^{\frac{8}{10m}}$. 16. $(a^{\frac{p^2}{q^2}-1})^{\frac{q}{p-q}}$.

244. The value of a numerical expression affected with a fractional exponent may be found by first, if possible, extracting the root indicated by the denominator, and then raising the result to the power indicated by the numerator.

Ex. Find the value of $(-8)^{\frac{2}{3}}$.

By § 243, $(-8)^{\frac{2}{3}} = [(-8)^{\frac{1}{3}}]^2 = (\sqrt[3]{-8})^2 = (-2)^2 = 4.$

EXERCISE 100

Find the values of the following:

1. $27^{\frac{2}{3}}$. 5. $81^{-\frac{1}{2}}$. 9. $256^{-\frac{3}{4}}$. 13. $243^{-\frac{6}{5}}$.

2. $16^{\frac{3}{4}}$. 6. $(-32)^{\frac{4}{5}}$. 10. $(-512)^{-\frac{1}{3}}$. 14. $(-128)^{\frac{3}{7}}$.

3. $64^{\frac{3}{2}}$. 7. $36^{-\frac{3}{2}}$. 11. $9^{\frac{7}{2}}$. 15. $729^{-\frac{5}{6}}$.

4. $64^{\frac{7}{6}}$. 8. $(-216)^{\frac{2}{3}}$. 12. $(-8)^{\frac{10}{3}}$. 16. $512^{\frac{5}{9}}$.

245. We will now show how to prove the result

$$(ab)^n = a^nb^n,$$

for any fractional or negative value of n.

The proof of this result in the case where n is any positive integer, was given in § 94.

I. Let $n = \frac{p}{q}$, where p and q are any positive integers.

By § 243, $$[(ab)^{\frac{p}{q}}]^q = (ab)^p = a^pb^p \text{ (§ 94)}. \qquad (1)$$

By § 94, $$(a^{\frac{p}{q}}b^{\frac{p}{q}})^q = (a^{\frac{p}{q}})^q(b^{\frac{p}{q}})^q = a^pb^p. \qquad (2)$$

From (1) and (2), $[(ab)^{\frac{p}{q}}]^q = (a^{\frac{p}{q}}b^{\frac{p}{q}})^q.$

Taking the qth root of both members, we have

$$(ab)^{\frac{p}{q}} = a^{\frac{p}{q}} b^{\frac{p}{q}}.$$

II. Let $n = -s$, where s is any positive integer or positive fraction.

Then, $(ab)^{-s} = \dfrac{1}{(ab)^s} = \dfrac{1}{a^s b^s}$ (§§ 94, or 245, I) $= a^{-s}b^{-s}$.

EXERCISE 101

Find the values of the following:

1. $(a^{\frac{1}{3}}b^{-2})^5$.
2. $(m^{-1}n^{-\frac{5}{6}})^{-3}$.
3. $(x^{-\frac{2}{3}}y^{\frac{6}{5}})^{\frac{1}{4}}$.
4. $(a^{\frac{1}{4}}x^{\frac{3}{2}})^{-\frac{2}{3}}$.
5. $(n^{-3}\sqrt{x^5})^{-4}$.
6. $(\sqrt[4]{a^3}\sqrt[5]{b^{-8}})^{\frac{5}{6}}$.

MISCELLANEOUS EXAMPLES

EXERCISE 102

Square the following by the rule of § 97:

1. $3\,a^{\frac{1}{3}} + 4\,b^{-\frac{3}{4}}$.

2. $5\,m^{-2}n^4 - 8\,m^2n^{-4}$.

3. Square $a^2b^{-\frac{1}{3}} - 2\,a^{\frac{1}{2}} - a^{-1}b^{\frac{1}{3}}$ by the rule of § 204.

4. Expand $(4\,x^{\frac{1}{3}}y^{-\frac{3}{4}} + 7\,z^{-2})(4\,x^{\frac{1}{3}}y^{-\frac{3}{4}} - 7\,z^{-2})$ by the rule of § 98.

Find the value of:

5. $\dfrac{25\,a^{-6} - 49\,m^{\frac{3}{2}}}{5\,a^{-3} - 7\,m^{\frac{3}{4}}}$, by the rule of § 101.

6. $\dfrac{8\,x^2 + 27\,y^{-\frac{5}{2}}}{2\,x^{\frac{2}{3}} + 3\,y^{-\frac{5}{6}}}$, by the rule of § 102.

7. $\dfrac{x^n - x^{-n}}{x^{\frac{n}{3}} - x^{-\frac{n}{3}}}$.

8. $\dfrac{a^{\frac{8}{5}} - b^{-\frac{2}{3}}}{a^{\frac{2}{5}} + b^{-\frac{1}{6}}}$, by the rule of § 103.

9. $(3\,x^{\frac{4}{3}} - 4\,y^{-\frac{5}{2}})^3$.

10. $(a^{-2}b^3 + 2\,a^3b^{-2})^3$.

Find the square roots of the following:

11. $16\,a^{-6}m^{\frac{1}{4}}$.

12. $49\,x^{\frac{6}{5}}yz^{-\frac{7}{3}}$.

13. $\dfrac{a^{7}m^{-\frac{4}{3}}}{4\,b^{\frac{5}{3}}n^{-3}}$.

14. $9\,x^{\frac{3}{2}}-6\,x^{\frac{3}{4}}+25-8\,x^{-\frac{3}{4}}+16\,x^{-\frac{3}{2}}$.

15. $4\,a^{-\frac{8}{5}}+20\,a^{-\frac{6}{5}}+21\,a^{-\frac{4}{5}}-10\,a^{-\frac{2}{5}}+1$.

16. $a^{\frac{4}{3}}b^{-3}-6\,a^{\frac{2}{3}}b^{-2}+5\,b^{-1}+12\,a^{-\frac{2}{3}}+4\,a^{-\frac{4}{3}}b$.

Find the cube roots of the following:

17. $8\,x^{5}y^{-6}$.

18. $-64\,a^{-4}b^{\frac{3}{5}}c^{-\frac{1}{4}}$.

19. $\dfrac{27\,m^{-\frac{6}{5}}n}{x^{\frac{1}{3}}y^{-\frac{5}{2}}}$.

20. $27\,x^{\frac{5}{2}}+54\,x^{\frac{5}{3}}y^{-\frac{2}{3}}+36\,x^{\frac{5}{6}}y^{-\frac{4}{3}}+8\,y^{-2}$.

21. $x^{\frac{1}{2}}-6\,x^{\frac{1}{6}}+21\,x^{-\frac{1}{6}}-44\,x^{-\frac{1}{2}}+63\,x^{-\frac{5}{6}}-54\,x^{-\frac{7}{6}}+27\,x^{-\frac{3}{2}}$.

Simplify the following, expressing all the results with positive exponents:

22. $[\sqrt[3]{(x^{\frac{5}{4}}y^{-2})}\div\sqrt[5]{(x^{-\frac{2}{3}}y^{4})}]^{\frac{10}{11}}$.

23. $\dfrac{\sqrt[10]{a^{9}}}{\sqrt[6]{b^{5}}\sqrt[9]{c}}\times\dfrac{\sqrt[4]{b^{3}}\sqrt[12]{c^{7}}}{\sqrt[15]{a^{8}}}$.

24. $[a^{n-1}\times(a^{-1})^{n+1}]\times[(a^{-n})^{-1}\times(a^{n-2})^{-1}]$.

25. $\left(x^{\frac{p+q}{q}}\times x^{\frac{q}{p-q}}\right)^{q}$.

26. $\left(\dfrac{a^{m+n}}{a^{m-n}}\right)^{2m}\left(\dfrac{a^{2m}}{a^{2n}}\right)^{m-n}$.

27. $\left(a^{\frac{n+1}{n-1}}\div a^{\frac{n-1}{n+1}}\right)^{\frac{n-1}{2n}}$.

28. $\left(x^{\frac{1}{m^{2}}}+x^{\frac{1}{n^{2}}}\right)^{\frac{mn}{m+n}}\div x^{-\frac{1}{n}}$.

29. $\left[\sqrt[2q]{\left(x^{\frac{p-q}{p+q}}\right)}\right]^{\frac{p}{p-q}-1}$.

30. $\dfrac{x^{\frac{1}{3}}+y^{\frac{1}{3}}}{x^{\frac{1}{3}}-y^{\frac{1}{3}}}-\dfrac{x+y}{x-y}$.

31. $\dfrac{a+b}{a^{\frac{1}{2}}+b^{\frac{1}{2}}}-\dfrac{a^{\frac{3}{2}}+b^{\frac{3}{2}}}{a-b}$.

32. $\dfrac{a^{\frac{1}{2}}+b^{\frac{1}{2}}}{a^{\frac{1}{2}}-b^{\frac{1}{2}}}+\dfrac{a^{-\frac{1}{2}}+b^{-\frac{1}{2}}}{a^{-\frac{1}{2}}-b^{-\frac{1}{2}}}$.

33. $\dfrac{x^{\frac{3m}{2n}}-1}{x^{\frac{m}{2n}}+1}+\dfrac{x^{\frac{3m}{2n}}+1}{x^{\frac{m}{2n}}-1}$.

34. $\dfrac{a^{\frac{1}{3}}+b^{\frac{1}{3}}}{a^{-\frac{1}{3}}-b^{-\frac{1}{3}}}\times\dfrac{a^{-\frac{1}{3}}+b^{-\frac{1}{3}}}{a^{\frac{1}{3}}-b^{\frac{1}{3}}}+1$.

35. $\dfrac{a^{\frac{1}{3}}+2\,b^{\frac{1}{3}}}{a^{\frac{1}{3}}-2\,b^{\frac{1}{3}}}-\dfrac{7\,a^{\frac{1}{3}}b^{\frac{1}{3}}+6\,b^{\frac{2}{3}}}{a^{\frac{2}{3}}+a^{\frac{1}{3}}b^{\frac{1}{3}}-6\,b^{\frac{2}{3}}}$.

XVIII. SURDS

246. A **Surd** is the indicated root of a number, or expression, which is not a perfect power of the degree denoted by the index of the radical sign; as $\sqrt{2}$, $\sqrt[3]{5}$, or $\sqrt[4]{x+y}$.

247. A monomial is said to be *rational* when it is rational and integral (§ 63), or else a fraction whose terms are rational and integral.

A polynomial is said to be rational when each of its terms is rational.

An expression is said to be *irrational* when it involves surds; as $2+\sqrt{3}$, or $\sqrt{a+1}-\sqrt{a}$.

248. A *rational number* is a positive or negative integer, or a positive or negative fraction.

An *irrational number* is a numerical expression involving surds; as $\sqrt[3]{3}$, or $2+\sqrt{5}$.

249. If a surd is in the form $b\sqrt[n]{a}$, b is called the *coefficient* of the surd, and n the *index*.

250. The *degree* of a surd is denoted by its index; thus, $\sqrt[3]{5}$ is a surd of the third degree.

A *quadratic surd* is a surd of the second degree.

REDUCTION OF A SURD TO ITS SIMPLEST FORM

251. A surd is said to be in its *simplest form* when the expression under the radical sign is rational and integral (§ 63), is not a perfect power of the degree denoted by any factor of the index of the surd, and has no factor which is a perfect power of the same degree as the surd.

252. Case I. *When the expression under the radical sign is a perfect power of the degree denoted by a factor of the index.*

Ex. Reduce $\sqrt[6]{8}$ to its simplest form.

We have, $\sqrt[6]{8}=\sqrt[6]{2^3}=2^{\frac{3}{6}}$ (§ 237) $=2^{\frac{1}{2}}=\sqrt{2}.$

EXERCISE 103

Reduce the following to their simplest forms:

1. $\sqrt[4]{25}$.	5. $\sqrt[16]{49}$.	9. $\sqrt[10]{243}$.	13. $\sqrt[12]{216\,a^9x^3}$.
2. $\sqrt[6]{16}$.	6. $\sqrt[16]{81}$.	10. $\sqrt[15]{343}$.	14. $\sqrt[12]{64\,a^6b^{18}}$.
3. $\sqrt[8]{121}$.	7. $\sqrt[14]{64}$.	11. $\sqrt[4]{144\,x^6y^8}$.	15. $\sqrt[18]{8\,a^{18}b^{15}}$.
4. $\sqrt[9]{125}$.	8. $\sqrt[10]{81}$.	12. $\sqrt[6]{27\,n^6x^{12}}$.	16. $\sqrt[12]{625\,x^{12}y^8}$.

253. CASE II. *When the expression under the radical sign is rational and integral, and has a factor which is a perfect power of the same degree as the surd.*

1. Reduce $\sqrt[3]{54}$ to its simplest form.

We have, $\sqrt[3]{54} = \sqrt[3]{27 \times 2} = \sqrt[3]{27} \times \sqrt[3]{2}$ (§ 234) $= 3\sqrt[3]{2}$.

2. Reduce $\sqrt{3\,a^3b - 12\,a^2b^2 + 12\,ab^3}$ tc its simplest form.

$$\sqrt{3\,a^3b - 12\,a^2b^2 + 12\,ab^3} = \sqrt{(a^2 - 4\,ab + 4\,b^2)3\,ab}$$

$$= \sqrt{a^2 - 4\,ab + 4\,b^2}\sqrt{3\,ab} = (a - 2\,b)\sqrt{3\,ab}.$$

We then have the following rule:

Resolve the expression under the radical sign into two factors, the first of which contains all the factors which are perfect powers of the same degree as the surd.

Extract the required root of the first factor, and multiply the result by the indicated root of the second.

If the expression under the radical sign has a numerical factor which cannot be readily factored by inspection, it is convenient to resolve it into its prime factors.

3. Reduce $\sqrt[3]{1944}$ to its simplest form.

$$\sqrt[3]{1944} = \sqrt[3]{2^3 \times 3^5} = \sqrt[3]{2^3 \times 3^3} \times \sqrt[3]{3^2} = 2 \times 3 \times \sqrt[3]{9} = 6\sqrt[3]{9}.$$

4. Reduce $\sqrt{125 \times 147}$ to its simplest form.

$$\sqrt{125 \times 147} = \sqrt{5^3 \times 3 \times 7^2} = \sqrt{5^2 \times 7^2} \times \sqrt{5 \times 3} = 5 \times 7 \times \sqrt{15} = 35\sqrt{15}.$$

EXERCISE 104

Reduce the following to their simplest forms:

1. $\sqrt{90}$.
2. $\sqrt{72}$.
3. $\sqrt{96}$.
4. $\sqrt{75}$.
5. $\sqrt[3]{56}$.
6. $7\sqrt{147}$.
7. $9\sqrt[3]{81}$.
8. $\sqrt[4]{48}$.
9. $\sqrt[6]{192}$.
10. $\sqrt[3]{432}$.
11. $\sqrt[7]{256}$.
12. $\sqrt{500\,a^4b^7}$.
13. $\sqrt{242\,x^3y^5}$.
14. $\sqrt[3]{500\,a^3b^8c^7}$.
15. $\sqrt[4]{162\,m^7n^{10}}$.
16. $\sqrt[5]{160\,x^3y^5z^8}$.
17. $\sqrt{75\,x^5y^3-100\,x^6y^7}$.
18. $\sqrt[3]{128\,a^5b^3+320\,a^3b^4}$
19. $\sqrt{(3\,x+2\,y)(9\,x^2-4\,y^2)}$.
20. $\sqrt{3\,a^3-24\,a^2+48\,a}$.
21. $\sqrt{18\,a^3b+60\,a^2b^2+50\,ab^3}$.
22. $\sqrt{(2\,x^2+x-15)(2\,x^2-19\,x+35)}$.
23. $\sqrt{896}$.
24. $\sqrt{2268}$.
25. $\sqrt{5145}$.
26. $\sqrt{98\times 196}$.
27. $\sqrt{432\times 504}$.
28. $\sqrt[3]{1372}$.
29. $\sqrt[3]{7875}$.
30. $\sqrt[3]{375\times 405}$.
31. $\sqrt[3]{54\times 63\times 336}$.
32. $\sqrt{63\,xy^2\times 175\,yz^2\times 875\,zx^2}$.

254. CASE III. *When the expression under the radical sign is a fraction.*

In this case, we *multiply both terms of the fraction by such an expression as will make the denominator a perfect power of the same degree as the surd,* and then proceed as in § 253.

Ex. Reduce $\sqrt{\dfrac{9}{8\,a^3}}$ to its simplest form.

Multiplying both terms of the fraction by $2\,a$, we have

$$\sqrt{\frac{9}{8\,a^3}}=\sqrt{\frac{9\times 2\,a}{16\,a^4}}=\sqrt{\frac{9}{16\,a^4}\times 2\,a}=\sqrt{\frac{9}{16\,a^4}}\times\sqrt{2\,a}=\frac{3}{4\,a^2}\sqrt{2\,a}.$$

EXERCISE 105

Reduce the following to their simplest forms:

1. $\sqrt{\frac{5}{2}}$.
2. $\sqrt{\frac{9}{5}}$.
3. $\sqrt{\frac{25}{8}}$.
4. $\sqrt{\frac{17}{18}}$.
5. $\sqrt{\frac{28}{27}}$.

6. $\sqrt{\frac{19}{72}}$.
7. $\sqrt[3]{\frac{4}{5}}$.
8. $\sqrt[3]{\frac{1}{9}}$.
9. $\sqrt[3]{\frac{13}{32}}$.
10. $\sqrt[3]{\frac{27}{4}}$.
11. $\sqrt[4]{\frac{3}{2}}$.
12. $\sqrt[4]{\frac{10}{9}}$.
13. $\sqrt[5]{\frac{2}{3}}$.
14. $\sqrt[5]{\frac{21}{8}}$.
15. $\sqrt{\frac{6}{7x}}$.
16. $\sqrt{\frac{23}{32a^5}}$.
17. $\sqrt{\frac{11a^6x^5}{20b^2y^3}}$.
18. $\sqrt[3]{\frac{7}{6m^2}}$.
19. $\sqrt[3]{\frac{8a^3}{25b}}$.
20. $\sqrt[3]{\frac{5x^4}{36y^3z^5}}$.
21. $\sqrt{\frac{2a}{2a+3b}}$.
22. $\frac{x^3}{x^2-5x+6}\sqrt{\frac{3x^2-18x+27}{x^3}}$.

255. To Introduce the Coefficient of a Surd under the Radical Sign.

The coefficient of a surd may be introduced under the radical sign by raising it to the power denoted by the index.

Ex. Introduce the coefficient of $2\sqrt[3]{3}$ under the radical sign.

$$2\sqrt[3]{3}=\sqrt[3]{8}\times\sqrt[3]{3}=\sqrt[3]{8\times 3}\ (\S\,234)=\sqrt[3]{24}.$$

A rational expression (§ 247) may be expressed in the form of a surd of any degree by raising it to the power denoted by the index, and writing the result under the corresponding radical sign.

EXERCISE 106

Introduce the coefficients of the following under the radical signs:

1. $3\sqrt{7}$.
2. $6\sqrt{6}$.
3. $4\sqrt[3]{5}$.
4. $5\sqrt[3]{7}$.
5. $4\sqrt[4]{5}$.
6. $2\sqrt[5]{8}$.
7. $2\sqrt[6]{3}$.
8. $9x\sqrt{2y}$.
9. $10a^3b^2\sqrt{6ab}$.
10. $6xy^2\sqrt[3]{4z^2}$.
11. $5an^3\sqrt[4]{2a^3n}$.
12. $3a^2b^3\sqrt[5]{3a^4b^5}$.
13. $(a-b)\sqrt{\frac{a+b}{a-b}}$.
14. $(2n+1)\sqrt{\frac{1}{4n^2-1}}$.
15. $\frac{a-1}{a+1}\sqrt{\frac{a^2+3a+2}{a^2-4a+3}}$.
16. $\frac{x^2-1}{x^2-2}\sqrt{2-\frac{(x-2)^2}{(x-1)^2}}$.

ADDITION AND SUBTRACTION OF SURDS

256. **Similar Surds** are surds which do not differ at all, or differ only in their coefficients; as $2\sqrt[3]{ax^2}$ and $3\sqrt[3]{ax^2}$.

Dissimilar Surds are surds which are not similar.

257. To add or subtract *similar surds* (§ 256), add or subtract their coefficients, and multiply the result by their common surd part.

1. Required the sum of $\sqrt{20}$ and $\sqrt{45}$.

Reducing each surd to its simplest form (§ 253),

$$\sqrt{20}+\sqrt{45}=\sqrt{4\times 5}+\sqrt{9\times 5}=2\sqrt{5}+3\sqrt{5}=5\sqrt{5}.$$

2. Simplify $\sqrt{\frac{1}{2}}+\sqrt{\frac{2}{3}}-\sqrt{\frac{9}{8}}$.

$$\sqrt{\frac{1}{2}}+\sqrt{\frac{2}{3}}-\sqrt{\frac{9}{8}}=\sqrt{\frac{1}{4}\times 2}+\sqrt{\frac{1}{9}\times 6}-\sqrt{\frac{9}{16}\times 2}$$

$$=\frac{1}{2}\sqrt{2}+\frac{1}{3}\sqrt{6}-\frac{3}{4}\sqrt{2}=\frac{1}{3}\sqrt{6}-\frac{1}{4}\sqrt{2}.$$

We then have the following rule:

Reduce each surd to its simplest form.

Add or subtract the similar surds, and indicate the addition or subtraction of the dissimilar.

EXERCISE 107

Simplify the following:

1. $\sqrt{8}+\sqrt{32}$.
2. $\sqrt{28}-\sqrt{63}$.
3. $\sqrt{300}-\sqrt{147}$.
4. $\sqrt[3]{2}+\sqrt[3]{128}$.
5. $\sqrt[3]{135}-\sqrt[3]{40}$.
6. $\sqrt[4]{80}+\sqrt[4]{405}$.
7. $\sqrt{3}+\sqrt{192}-\sqrt{243}$.
8. $\sqrt{250}-\sqrt{90}-\sqrt{176}$.
9. $\sqrt{\frac{4}{3}}+\sqrt{\frac{25}{12}}$.
10. $\sqrt{\frac{27}{2}}+\sqrt{\frac{8}{3}}$.
11. $\sqrt[3]{\frac{7}{4}}-\sqrt[3]{\frac{7}{108}}$.
12. $\sqrt{99}-\sqrt{275}+\sqrt{396}$.
13. $\sqrt[3]{56}+\sqrt[3]{189}+\sqrt[3]{162}$.
14. $\sqrt{\frac{5}{18}}-\sqrt{\frac{32}{5}}+\sqrt{\frac{40}{9}}$.
15. $\sqrt{\frac{15}{16}}+\sqrt{\frac{27}{20}}-\sqrt{\frac{48}{5}}$.
16. $\sqrt{72x^5}-x\sqrt{98x^3}+x^2\sqrt{200x}$.

17. $a\sqrt[3]{80\,a^2b^7}+ab\sqrt[3]{270\,a^2b^4}+b^2\sqrt[3]{640\,a^5b}$.

18. $\sqrt{27\,x^5+36\,x^4y}+\sqrt{48\,xy^2+64\,y^3}$.

19. $\sqrt{\frac{7}{3}}-\sqrt{\frac{12}{7}}+\sqrt{\frac{4}{21}}$.

20. $\sqrt[3]{\frac{5}{4}}+\sqrt[3]{\frac{16}{25}}+\sqrt[3]{\frac{1}{100}}$.

21. $\sqrt[3]{128}+\sqrt[3]{250}-\sqrt[3]{432}-\sqrt[3]{88}$.

22. $\sqrt{50\,a^2}+\sqrt{72\,b^2}-\sqrt{50\,a^2+120\,ab+72\,b^2}$.

23. $\sqrt[4]{96}+\sqrt[4]{486}-\sqrt[4]{6}$.

24. $\sqrt{294}-\sqrt{216}+\sqrt{405}-\sqrt{600}$.

25. $\sqrt{52\,ab^2}-b\sqrt{117\,a}-\sqrt{126\,a^5b}+a\sqrt{56\,a^3b}$.

26. $\sqrt{\frac{5}{4}}+\sqrt{\frac{9}{5}}-\sqrt{\frac{1}{20}}-\sqrt{\frac{1}{72}}$.

27. $\sqrt{\frac{56}{25}}-\sqrt{\frac{8}{20}}-\sqrt{\frac{2}{7}}+\sqrt{\frac{12}{5}}$.

28. $\sqrt{50\,x^3+40\,x^2+8\,x}-\sqrt{32\,x^3-48\,x^2+18\,x}$.

29. $\sqrt{125\,x^2-150\,xy+45\,y^2}+\sqrt{5\,x^2+60\,xy+180\,y^2}$.

30. $(x+y)\sqrt{\dfrac{x+y}{x-y}}+(x-y)\sqrt{\dfrac{x-y}{x+y}}-\dfrac{2(x^2+y^2)}{x^2-y^2}\sqrt{x^2-y^2}$.

TO REDUCE SURDS OF DIFFERENT DEGREES TO EQUIVALENT SURDS OF THE SAME DEGREE

258. *Ex.* Reduce $\sqrt{2}$, $\sqrt[3]{3}$, and $\sqrt[4]{5}$ to equivalent surds of the same degree.

By § 237, $\sqrt{2}=2^{\frac{1}{2}}=2^{\frac{6}{12}}=\sqrt[12]{2^6}=\sqrt[12]{64}$.

$$\sqrt[3]{3}=3^{\frac{1}{3}}=3^{\frac{4}{12}}=\sqrt[12]{3^4}=\sqrt[12]{81}.$$

$$\sqrt[4]{5}=5^{\frac{1}{4}}=5^{\frac{3}{12}}=\sqrt[12]{5^3}=\sqrt[12]{125}.$$

We then have the following rule:

Express the surds with fractional exponents, reduce these to their lowest common denominator, and express the resulting expressions with radical signs.

The relative magnitudes of surds may be determined by reducing them, if necessary, to equivalent surds of the same degree.

Thus, in the above example, $\sqrt[12]{125}$ is greater than $\sqrt[12]{81}$, and $\sqrt[12]{81}$ than $\sqrt[12]{64}$.

Then, $\sqrt[4]{5}$ is greater than $\sqrt[3]{3}$, and $\sqrt[3]{3}$ than $\sqrt{2}$.

EXERCISE 108

Reduce the following to equivalent surds of the same degree:

1. $\sqrt{2}$ and $\sqrt[3]{7}$.
2. $\sqrt{3}$ and $\sqrt[5]{4}$.
3. $\sqrt[3]{2}$ and $\sqrt[5]{3}$.
4. $\sqrt[3]{4}$ and $\sqrt[4]{5}$.
5. $\sqrt{3}$ and $\sqrt[7]{6}$.
6. $\sqrt[4]{ab}$, $\sqrt[7]{bc}$, and $\sqrt[14]{ca}$.
7. $\sqrt{2a}$, $\sqrt[3]{3b}$, and $\sqrt[4]{5c}$.
8. $\sqrt[3]{2x^2}$, $\sqrt[6]{5y^3}$, and $\sqrt[8]{6z^3}$.
9. $\sqrt[6]{a+1}$ and $\sqrt[9]{a-1}$.
10. $\sqrt[12]{x-y}$ and $\sqrt[9]{x+y}$.
11. Which is the greater, $\sqrt{6}$ or $\sqrt[3]{14}$?
12. Which is the greater, $\sqrt{2}$ or $\sqrt[5]{5}$?
13. Which is the greater, $\sqrt[3]{3}$ or $\sqrt[5]{7}$?
14. Arrange in order of magnitude $\sqrt{2}$, $\sqrt[7]{13}$, and $\sqrt[14]{31}$.
15. Arrange in order of magnitude $\sqrt[3]{4}$, $\sqrt[4]{6}$, and $\sqrt[6]{15}$.
16. Arrange in order of magnitude $\sqrt[3]{2}$, $\sqrt[6]{3}$, and $\sqrt[9]{10}$.

MULTIPLICATION OF SURDS

259. 1. Multiply $\sqrt{6}$ by $\sqrt{15}$.

By § 234, $\sqrt{6} \times \sqrt{15} = \sqrt{6 \times 15} = \sqrt{2 \times 3 \times 3 \times 5} = \sqrt{3^2 \times 2 \times 5} = 3\sqrt{10}$.

2. Multiply $\sqrt{2a}$ by $\sqrt[3]{4a^2}$.

Reducing to equivalent surds of the same degree (§ 258),

$$\sqrt{2a} \times \sqrt[3]{4a^2} = (2a)^{\frac{1}{2}} \times (4a^2)^{\frac{1}{3}} = (2a)^{\frac{3}{6}} \times (4a^2)^{\frac{2}{6}} = \sqrt[6]{(2a)^3} \times \sqrt[6]{(4a^2)^2}$$

$$= \sqrt[6]{2^3 a^3 \times 2^4 a^4} = \sqrt[6]{2^6 a^6 \times 2a} = 2a\sqrt[6]{2a}.$$

We then have the following rule:

To multiply together two or more surds, reduce them, if necessary, to surds of the same degree.

Multiply together the expressions under the radical signs, and write the result under the common radical sign.

The result should be reduced to its simplest form.

3. Multiply $\sqrt{5}$ by $\sqrt[6]{5}$.

By § 237, $\sqrt{5} = 5^{\frac{1}{2}} = 5^{\frac{3}{6}} = \sqrt[6]{5^3}$.

Then, $\sqrt{5} \times \sqrt[6]{5} = \sqrt[6]{5^3} \times \sqrt[6]{5} = \sqrt[6]{5^4} = 5^{\frac{4}{6}} = 5^{\frac{2}{3}} = \sqrt[3]{5^2} = \sqrt[3]{25}$.

4. Multiply $2\sqrt{3} + 3\sqrt{2}$ by $3\sqrt{3} - \sqrt{2}$.

$$\begin{array}{l} 2\sqrt{3} + 3\sqrt{2} \\ \underline{3\sqrt{3} - \quad \sqrt{2}} \\ \quad 18 + 9\sqrt{6} \\ \quad \underline{\quad\quad - 2\sqrt{6} - 6} \\ \quad 18 + 7\sqrt{6} - 6 = 12 + 7\sqrt{6}. \end{array}$$

To multiply a surd of the second degree by itself simply removes the radical sign; thus, $\sqrt{3} \times \sqrt{3} = 3$.

5. Multiply $3\sqrt{1+x} - 4\sqrt{x}$ by $\sqrt{1+x} + 2\sqrt{x}$.

$$\begin{array}{l} 3\sqrt{1+x} - 4\sqrt{x} \\ \underline{\ \sqrt{1+x} + 2\sqrt{x}} \\ 3(1+x) - 4\sqrt{x+x^2} \\ \underline{\quad\quad\quad + 6\sqrt{x+x^2} - 8x} \\ 3(1+x) + 2\sqrt{x+x^2} - 8x = 3 - 5x + 2\sqrt{x+x^2}. \end{array}$$

EXERCISE 109

Multiply the following:

1. $\sqrt{5}$ by $\sqrt{20}$.
2. $\sqrt[3]{49x^2}$ by $\sqrt[3]{7x}$.
3. $\sqrt{15}$ by $\sqrt{27}$.
4. $\sqrt{18}$ by $\sqrt{42}$.
5. $\sqrt{108}$ by $\sqrt{192}$.
6. $\sqrt[3]{72}$ by $\sqrt[3]{81}$.
7. $\sqrt{55xy}$ by $\sqrt{66yz}$.
8. $\sqrt[3]{35}$ by $\sqrt[3]{75}$.
9. $\sqrt[3]{84}$ by $\sqrt[3]{180}$.
10. $\sqrt{\frac{14}{3}}$ by $\sqrt{\frac{35}{27}}$.
11. $\sqrt[4]{98}$ by $\sqrt[4]{343}$.
12. $\sqrt[5]{63}$ by $\sqrt[5]{135}$.
13. $\sqrt{6ab}$ by $\sqrt[4]{2ac}$.
14. $\sqrt{6}$ by $\sqrt[3]{9}$.
15. $\sqrt[3]{3xy}$ by $\sqrt[9]{7yz}$.
16. $\sqrt[3]{44}$ by $\sqrt[6]{12}$.
17. $\sqrt[4]{135}$ by $\sqrt[8]{45}$.
18. $\sqrt{20}$ by $\sqrt[10]{5}$.

19. $\sqrt[3]{5a^2}$ by $\sqrt[4]{125a^2}$.

20. $\sqrt[3]{9}$ by $\sqrt[5]{27}$.

21. $\sqrt{\frac{4}{27}}$ by $\sqrt[5]{\frac{27}{8}}$.

22. $\sqrt{\frac{3}{5}}$ by $\sqrt[6]{\frac{1}{45}}$.

23. $\sqrt[3]{xy}$, $\sqrt[6]{yz}$, and $\sqrt[12]{zx^3}$.

24. $\sqrt{20}$, $\sqrt[3]{25}$, and $\sqrt[6]{5}$.

25. $\sqrt[3]{4}$, $\sqrt[4]{6}$, and $\sqrt[6]{6}$.

26. $\sqrt{15}$, $\sqrt[6]{\frac{1}{27}}$, and $\sqrt[9]{\frac{1}{125}}$.

27. $6+3\sqrt{2}$ and $4+5\sqrt{2}$.

28. $4\sqrt{a}-3\sqrt{b}$ and $7\sqrt{a}+2\sqrt{b}$.

29. $2\sqrt{5}-8\sqrt{3}$ and $9\sqrt{5}-4\sqrt{3}$.

30. $5\sqrt{2}+6\sqrt{6}$ and $10\sqrt{2}-7\sqrt{6}$.

31. $2\sqrt[3]{9}+9\sqrt[3]{7}$ and $8\sqrt[3]{3}-3\sqrt[3]{49}$.

32. $4\sqrt{x}-\sqrt{y}+3\sqrt{z}$ and $4\sqrt{x}+\sqrt{y}-3\sqrt{z}$.

33. $3\sqrt{a+2}+4\sqrt{a-1}$ and $6\sqrt{a+2}+5\sqrt{a-1}$.

34. $\sqrt{2}+\sqrt{5}+\sqrt{7}$ and $\sqrt{2}-\sqrt{5}-\sqrt{7}$.

35. $4\sqrt{\frac{3}{2}}-3\sqrt{\frac{4}{27}}$ and $2\sqrt{\frac{3}{2}}-9\sqrt{\frac{4}{27}}$.

36. $3\sqrt{3}+2\sqrt{6}-4\sqrt{8}$ and $3\sqrt{3}-2\sqrt{6}+4\sqrt{8}$.

37. $6\sqrt{5}-5\sqrt{7}-\sqrt{10}$ and $6\sqrt{5}+5\sqrt{7}+\sqrt{10}$.

38. $8\sqrt{12}+7\sqrt{20}-4\sqrt{24}$ and $5\sqrt{3}-3\sqrt{5}+2\sqrt{6}$.

39. $6\sqrt{\frac{2}{3}}+8\sqrt{\frac{3}{5}}+11\sqrt{\frac{5}{2}}$ and $3\sqrt{\frac{2}{3}}-4\sqrt{\frac{3}{5}}-5\sqrt{\frac{5}{2}}$.

DIVISION OF MONOMIAL SURDS

260. By § 234, $\sqrt[n]{ab}=\sqrt[n]{a}\times\sqrt[n]{b}$.

Whence, $\dfrac{\sqrt[n]{ab}}{\sqrt[n]{a}}=\sqrt[n]{b}$.

We then have the following rule:

To divide one monomial surd by another, reduce them, if necessary, to surds of the same degree.

Divide the expression under the radical sign in the dividend by the expression under the radical sign in the divisor, and write the result under the common radical sign.

The result should be reduced to its simplest form.

1. Divide $\sqrt[3]{405}$ by $\sqrt[3]{5}$.

We have, $\dfrac{\sqrt[3]{405}}{\sqrt[3]{5}} = \sqrt[3]{\dfrac{405}{5}} = \sqrt[3]{81} = \sqrt[3]{27 \times 3} = 3\sqrt[3]{3}.$

2. Divide $\sqrt[3]{4}$ by $\sqrt{6}$.

Reducing to surds of the same degree (§ 258),

$$\frac{\sqrt[3]{4}}{\sqrt{6}} = \frac{4^{\frac{1}{3}}}{6^{\frac{1}{2}}} = \frac{(2^2)^{\frac{2}{6}}}{(2 \times 3)^{\frac{3}{6}}} = \frac{\sqrt[6]{2^4}}{\sqrt[6]{2^3 \times 3^3}} = \sqrt[6]{\frac{2^4}{2^3 \times 3^3}} = \sqrt[6]{\frac{2}{3^3}} = \sqrt[6]{\frac{2 \times 3^3}{3^6}} = \tfrac{1}{3}\sqrt[6]{54}.$$

3. Divide $\sqrt{10}$ by $\sqrt[6]{40}$.

We have, $\sqrt{10} = 10^{\frac{1}{2}} = 10^{\frac{3}{6}} = \sqrt[6]{10^3} = \sqrt[6]{(2 \times 5)^3}.$

Then, $\dfrac{\sqrt{10}}{\sqrt[6]{40}} = \sqrt[6]{\dfrac{2^3 \times 5^3}{2^3 \times 5}} = \sqrt[6]{5^2} = 5^{\frac{2}{6}} = 5^{\frac{1}{3}} = \sqrt[3]{5}.$

EXERCISE 110

Divide the following:

1. $\sqrt{90}$ by $\sqrt{5}$.
2. $\sqrt{24}$ by $\sqrt{18}$.
3. $\sqrt{70}$ by $\sqrt{63}$.
4. $\sqrt[3]{144}$ by $\sqrt[3]{9}$.
5. $\sqrt[3]{3}$ by $\sqrt[3]{192}$.
6. $\sqrt[3]{48}$ by $\sqrt[3]{56}$.
7. $\sqrt[4]{32}$ by $\sqrt[3]{2}$.
8. $\sqrt[4]{486}$ by $\sqrt[4]{2}$.
9. $\sqrt{7}$ by $\sqrt[3]{49}$.
10. $\sqrt[5]{42\,x^2}$ by $\sqrt[5]{56\,x^4}$.
11. $\sqrt[9]{686}$ by $\sqrt[3]{63}$.
12. $\sqrt{25\,a}$ by $\sqrt[6]{25\,a}$.
13. $\sqrt{\frac{52}{15}}$ by $\sqrt{\frac{65}{12}}$.
14. $\sqrt{1\frac{1}{27}}$ by $\sqrt{5\frac{5}{6}}$.
15. $\sqrt[3]{\frac{18}{25}}$ by $\sqrt[3]{\frac{4}{45}}$.
16. $\sqrt[5]{6\,ab^2}$ by $\sqrt[10]{96\,bc^3}$.
17. $\sqrt[5]{3\,a^3}$ by $\sqrt{2\,a}$.
18. $\sqrt{27\,x^3}$ by $\sqrt[3]{36\,x^2}$.
19. $\sqrt{\frac{5}{8}}$ by $\sqrt[5]{\frac{25}{16}}$.
20. $\sqrt[3]{\frac{49}{27}}$ by $\sqrt{\frac{7}{3}}$.
21. $\sqrt[3]{12\,a^3b^5}$ by $\sqrt[4]{4\,a^2b^4}$.
22. $\sqrt[12]{\frac{2}{243}}$ by $\sqrt[3]{\frac{4}{9}}$.
23. $\sqrt[5]{\frac{2}{27}}$ by $\sqrt[3]{\frac{8}{9}}$.
24. $\sqrt[3]{15\,nx^2}$ by $\sqrt[4]{405\,n^2x}$.

INVOLUTION OF SURDS

261. **1.** Raise $\sqrt[6]{12}$ to the third power.

$$(\sqrt[6]{12})^3 = (12^{\frac{1}{6}})^3 = 12^{\frac{3}{6}}\ (\S\ 243) = 12^{\frac{1}{2}} = \sqrt{12} = 2\sqrt{3}.$$

2. Raise $\sqrt[5]{2}$ to the fourth power.

$$(\sqrt[5]{2})^4 = (2^{\frac{1}{5}})^4 = 2^{\frac{4}{5}} = \sqrt[5]{2^4} = \sqrt[5]{16}.$$

Then, to raise a surd to any positive integral power,

If possible, divide the index of the surd by the exponent of the required power; otherwise, raise the expression under the radical sign to the required power.

The rules of §§ 97 and 98 should be used to find the value of any product which comes under them.

3. Expand $(\sqrt{6}-\sqrt{3})^2$.

By § 97, $$(\sqrt{6}-\sqrt{3})^2 = (\sqrt{6})^2 - 2\sqrt{6}\times\sqrt{3} + (\sqrt{3})^2$$
$$= 6 - 2\sqrt{3^2\times 2} + 3 = 9 - 6\sqrt{2}.$$

4. Expand $(4+\sqrt[6]{5})(4-\sqrt[6]{5})$.

By § 98, $(4+\sqrt[6]{5})(4-\sqrt[6]{5}) = 4^2 - (\sqrt[6]{5})^2 = 16 - \sqrt[3]{5}$, by the above rule.

EXERCISE 111

Find the values of the following:

1. $(\sqrt[7]{2})^4$.
2. $(\sqrt[6]{6})^2$.
3. $(\sqrt[5]{4x+3y})^2$.
4. $(\sqrt[8]{32})^2$.
5. $(\sqrt{2mn^2})^7$.
6. $(5\sqrt[12]{5x^7})^2$.
7. $(\sqrt[4]{a-b})^3$.
8. $(\sqrt[10]{72an^3})^5$.
9. $(\sqrt[9]{54a^2b})^3$.
10. $(\sqrt[8]{5})^9$.
11. $(3a^3\sqrt[12]{250a^4})^4$.
12. $(4\sqrt[15]{729})^3$.
13. $(7+2\sqrt{2})^2$.
14. $(4\sqrt{5}-5)^2$.
15. $(3\sqrt{6}+6\sqrt{3})^2$.
16. $(9\sqrt{7}-4\sqrt{11})^2$.
17. $(\sqrt{5x+2}-\sqrt{3x})^2$.
18. $(4\sqrt{a-b}+3\sqrt{a+b})^2$.
19. $(6+5\sqrt{2})(6-5\sqrt{2})$.
20. $(4\sqrt{a}+3\sqrt{a-1})(4\sqrt{a}-3\sqrt{a-1})$.
21. $(\sqrt{2x+y}+\sqrt{2x-y})(\sqrt{2x+y}-\sqrt{2x-y})$.
22. $(5\sqrt{3x+4}+4\sqrt{5x-2})(5\sqrt{3x+4}-4\sqrt{5x-2})$.
23. $\sqrt[3]{4+2\sqrt{3}}\times\sqrt[3]{4-2\sqrt{3}}$.
24. $(\sqrt[3]{4}+\sqrt[3]{9})(\sqrt[3]{4}-\sqrt[3]{9})$.
25. $\sqrt{3\sqrt{5}+2\sqrt{7}}\times\sqrt{3\sqrt{5}-2\sqrt{7}}$.
26. Expand $(2\sqrt{2}+\sqrt{6}-\sqrt{3})^2$, by the rule of § 204.

EVOLUTION OF SURDS

262. **1.** Extract the cube root of $\sqrt[5]{27x^3}$.

$$\sqrt[3]{(\sqrt[5]{27x^3})} = (\sqrt[5]{(3x)^3})^{\frac{1}{3}} = [(3x)^{\frac{3}{5}}]^{\frac{1}{3}} = (3x)^{\frac{1}{5}} = \sqrt[5]{3x}.$$

2. Extract the fifth root of $\sqrt[3]{6}$.

$$\sqrt[5]{(\sqrt[3]{6})} = (6^{\frac{1}{3}})^{\frac{1}{5}} = 6^{\frac{1}{15}} = \sqrt[15]{6}.$$

Then, to extract any root of a surd,

If possible, extract the required root of the expression under the radical sign; otherwise, multiply the index of the surd by the index of the required root.

If the surd has a coefficient which is not a perfect power of the degree denoted by the index of the required root, it should be introduced under the radical sign (§ 255) before applying the rule.

Thus, $$\sqrt[5]{(4\sqrt{2})} = \sqrt[5]{(\sqrt{32})} = \sqrt{2}.$$

EXERCISE 112

Find the values of the following:

1. $\sqrt{(\sqrt[6]{25})}$.
2. $\sqrt{(\sqrt[3]{13})}$.
3. $\sqrt[3]{(\sqrt[7]{8a^3b^6})}$.
4. $\sqrt[5]{(\sqrt[4]{243x^5})}$.
5. $\sqrt{(\sqrt[5]{9a^2+12a+4})}$.
6. $\sqrt[3]{(\sqrt[4]{49})}$.
7. $\sqrt[4]{(81\sqrt[3]{16})}$.
8. $\sqrt[5]{(2\sqrt[3]{3a^2b})}$.
9. $\sqrt[4]{(16a^4\sqrt[4]{3a})}$.
10. $\sqrt[7]{(2x\sqrt[5]{4x^2})}$.
11. $\sqrt[6]{(\sqrt{343})}$.
12. $\sqrt[3]{(2n^2\sqrt[5]{16n^2})}$.

REDUCTION OF A FRACTION WHOSE DENOMINATOR IS IRRATIONAL (§ 247) TO AN EQUIVALENT FRACTION HAVING A RATIONAL DENOMINATOR

263. Case I. *When the denominator is a monomial.*

The reduction may be effected by multiplying both terms of the fraction by a surd of the same degree as the denominator, having under its radical sign such an expression as will make the denominator of the resulting fraction rational.

Ex. Reduce $\frac{5}{\sqrt[3]{3a^2}}$ to an equivalent fraction having a rational denominator.

Multiplying both terms by $\sqrt[3]{9a}$, we have

$$\frac{5}{\sqrt[3]{3a^2}} = \frac{5\sqrt[3]{9a}}{\sqrt[3]{3a^2}\sqrt[3]{9a}} = \frac{5\sqrt[3]{9a}}{\sqrt[3]{27a^3}} = \frac{5\sqrt[3]{9a}}{3a}.$$

EXERCISE 113

Reduce each of the following to an equivalent fraction having a rational denominator:

1. $\frac{3}{\sqrt{5}}$.
2. $\frac{2}{\sqrt{12x^3y}}$.
3. $\frac{a^2}{\sqrt[3]{6a^2}}$.
4. $\frac{1}{\sqrt[3]{49x}}$.
5. $\frac{4}{\sqrt[4]{25}}$.
6. $\frac{6x^3y}{\sqrt[4]{8x^3yz^2}}$.
7. $\frac{9}{\sqrt[5]{27}}$.
8. $\frac{7}{\sqrt[6]{4a^2bc^4}}$.

264. Case II. *When the denominator is a binomial containing only surds of the second degree.*

1. Reduce $\frac{5-\sqrt{2}}{5+\sqrt{2}}$ to an equivalent fraction having a rational denominator.

Multiplying both terms by $5-\sqrt{2}$, we have

$$\frac{5-\sqrt{2}}{5+\sqrt{2}} = \frac{(5-\sqrt{2})^2}{(5+\sqrt{2})(5-\sqrt{2})} = \frac{25-10\sqrt{2}+2}{25-2} \text{ (§§ 97, 98)} = \frac{27-10\sqrt{2}}{23}.$$

2. Reduce $\frac{3\sqrt{a}-2\sqrt{a-b}}{2\sqrt{a}-3\sqrt{a-b}}$ to an equivalent fraction having a rational denominator.

Multiplying both terms by $2\sqrt{a}+3\sqrt{a-b}$,

$$\frac{3\sqrt{a}-2\sqrt{a-b}}{2\sqrt{a}-3\sqrt{a-b}} = \frac{(3\sqrt{a}-2\sqrt{a-b})(2\sqrt{a}+3\sqrt{a-b})}{(2\sqrt{a}-3\sqrt{a-b})(2\sqrt{a}+3\sqrt{a-b})}$$

$$= \frac{6a+5\sqrt{a}\sqrt{a-b}-6(a-b)}{4a-9(a-b)} = \frac{6b+5\sqrt{a^2-ab}}{9b-5a}.$$

We then have the following rule:

Multiply both terms of the fraction by the denominator with the sign between its terms reversed.

EXERCISE 114

Reduce each of the following to an equivalent fraction having a rational denominator:

1. $\dfrac{8}{\sqrt{6}+2}$.

2. $\dfrac{7}{5-3\sqrt{2}}$.

3. $\dfrac{m-\sqrt{n}}{m+\sqrt{n}}$.

4. $\dfrac{\sqrt{x}+\sqrt{y}}{\sqrt{x}-\sqrt{y}}$.

5. $\dfrac{\sqrt{10}-6\sqrt{2}}{\sqrt{10}+2\sqrt{2}}$.

6. $\dfrac{2\sqrt{7}+3\sqrt{3}}{2\sqrt{7}-3\sqrt{3}}$.

7. $\dfrac{3\sqrt{5}-\sqrt{3}}{4\sqrt{5}+5\sqrt{3}}$.

8. $\dfrac{3-\sqrt{a-3}}{4-\sqrt{a-3}}$.

9. $\dfrac{\sqrt{x}-\sqrt{x+y}}{\sqrt{x}+\sqrt{x+y}}$.

10. $\dfrac{\sqrt{9a^2-2}-3a}{\sqrt{9a^2-2}+3a}$.

11. $\dfrac{\sqrt{x^2+y^2}+y}{\sqrt{x^2+y^2}-y}$.

12. $\dfrac{2\sqrt{x-2}+\sqrt{x+2}}{2\sqrt{x-2}-\sqrt{x+2}}$.

13. $\dfrac{2}{\sqrt{\sqrt{11}+3}-\sqrt{\sqrt{11}-3}}$.

14. $\dfrac{\sqrt{x^2+y^2}+\sqrt{x^2-y^2}}{\sqrt{x^2+y^2}-\sqrt{x^2-y^2}}$.

15. $\dfrac{\sqrt{a+1}+2\sqrt{a-1}}{4\sqrt{a+1}+3\sqrt{a-1}}$.

265. If the denominator is a *trinomial*, containing only surds of the second degree, the fraction may be reduced to an equivalent fraction having a rational denominator by two applications of the rule of § 264.

Ex. Reduce $\dfrac{4-\sqrt{3}-\sqrt{7}}{4+\sqrt{3}-\sqrt{7}}$ to an equivalent fraction having a rational denominator.

Multiplying both terms by $4+\sqrt{3}+\sqrt{7}$, we have

$$\frac{4-\sqrt{3}-\sqrt{7}}{4+\sqrt{3}-\sqrt{7}}=\frac{(4-\sqrt{3}-\sqrt{7})(4+\sqrt{3}+\sqrt{7})}{(4+\sqrt{3}-\sqrt{7})(4+\sqrt{3}+\sqrt{7})}=\frac{4^2-(\sqrt{3}+\sqrt{7})^2}{(4+\sqrt{3})^2-(\sqrt{7})^2}\ (\S\ 98).$$

Then, $\dfrac{4-\sqrt{3}-\sqrt{7}}{4+\sqrt{3}-\sqrt{7}}=\dfrac{16-(10+2\sqrt{21})}{19+8\sqrt{3}-7}=\dfrac{6-2\sqrt{21}}{12+8\sqrt{3}}=\dfrac{3-\sqrt{21}}{6+4\sqrt{3}}.$

Multiplying both terms of the latter by $6-4\sqrt{3}$,

$$\frac{4-\sqrt{3}-\sqrt{7}}{4+\sqrt{3}-\sqrt{7}}=\frac{(3-\sqrt{21})(6-4\sqrt{3})}{6^2-(4\sqrt{3})^2}$$

$$=\frac{18-6\sqrt{21}-12\sqrt{3}+4\sqrt{63}}{-12}=\frac{-9+3\sqrt{21}+6\sqrt{3}-6\sqrt{7}}{6}.$$

The example may also be solved by multiplying both terms of the given fraction by $4-\sqrt{3}+\sqrt{7}$, or by $4-\sqrt{3}-\sqrt{7}$.

EXERCISE 115

Reduce each of the following to an equivalent fraction having a rational denominator:

1. $\dfrac{7}{2+\sqrt{2}+\sqrt{3}}$.

2. $\dfrac{6}{3+\sqrt{5}-\sqrt{2}}$.

3. $\dfrac{12}{\sqrt{5}-\sqrt{3}-\sqrt{2}}$.

4. $\dfrac{\sqrt{6}+\sqrt{3}-3\sqrt{2}}{\sqrt{6}-\sqrt{3}+3\sqrt{2}}$.

The reduction of a fraction having an irrational denominator to an equivalent fraction having a rational denominator, when the denominator is the sum of a rational expression and a surd of the nth degree, or of two surds of the nth degree, will be found in § 446.

266. The approximate value of a fraction whose denominator is irrational may be conveniently found by reducing it to an equivalent fraction with a rational denominator.

Ex. Find the approximate value of $\dfrac{1}{2-\sqrt{2}}$ to three places of decimals.

$$\frac{1}{2-\sqrt{2}}=\frac{2+\sqrt{2}}{(2-\sqrt{2})(2+\sqrt{2})}=\frac{2+\sqrt{2}}{4-2}=\frac{2+1.414\cdots}{2}=1.707\cdots.$$

EXERCISE 116

Find the values of the following to three places of decimals:

1. $\dfrac{4}{\sqrt{6}}$.

2. $\dfrac{5}{3-\sqrt{3}}$.

3. $\dfrac{1}{5+2\sqrt{7}}$.

4. $\dfrac{3}{\sqrt[3]{49}}$.

5. $\dfrac{23}{\sqrt{5}-3\sqrt{2}}$.

6. $\dfrac{\sqrt{7}-\sqrt{2}}{\sqrt{7}+\sqrt{2}}$.

7. $\dfrac{2\sqrt{6}+\sqrt{3}}{2\sqrt{6}-\sqrt{3}}$.

8. $\dfrac{4\sqrt{5}-5\sqrt{3}}{4\sqrt{5}+5\sqrt{3}}$.

9. $\dfrac{4\sqrt{7}+7\sqrt{3}}{3\sqrt{7}-5\sqrt{3}}$.

PROPERTIES OF QUADRATIC SURDS (§ 250)

267. *A quadratic surd cannot equal the sum of a rational expression and a quadratic surd.*

For, if possible, let $\quad \sqrt{a}=b+\sqrt{c},$

where b is a rational expression, and $\sqrt{a}$ and $\sqrt{c}$ quadratic surds.

Squaring both members, $a=b^2+2b\sqrt{c}+c,$

or,
$$2b\sqrt{c}=a-b^2-c.$$

Whence,
$$\sqrt{c}=\frac{a-b^2-c}{2b}.$$

That is, a quadratic surd equal to a rational expression.

But this is impossible; whence, $\sqrt{a}$ cannot equal $b+\sqrt{c}$.

268. *If $a+\sqrt{b}=c+\sqrt{d}$, where a and c are rational expressions, and $\sqrt{b}$ and $\sqrt{d}$ quadratic surds, then*

$$a=c, \text{ and } \sqrt{b}=\sqrt{d}.$$

If a does not equal c, let $a=c+x$; then, x is rational.

Substituting this value in the given equation,

$$c+x+\sqrt{b}=c+\sqrt{d}, \text{ or } x+\sqrt{b}=\sqrt{d}.$$

But this is impossible by § 267.

Then, $a=c$, and therefore $\sqrt{b}=\sqrt{d}$.

269. *If $\sqrt{a+\sqrt{b}}=\sqrt{x}+\sqrt{y}$, where a, b, x, and y are rational expressions, then $\sqrt{a-\sqrt{b}}=\sqrt{x}-\sqrt{y}$.*

Squaring both members of the given equation,

$$a+\sqrt{b}=x+2\sqrt{xy}+y.$$

Whence, by § 268, $a = x + y$,

and $\sqrt{b} = 2\sqrt{xy}$.

Subtracting, $a - \sqrt{b} = x - 2\sqrt{xy} + y$.

Extracting the square root of both members,

$$\sqrt{a - \sqrt{b}} = \sqrt{x} - \sqrt{y}.$$

270. Square Root of a Binomial Surd.

The preceding principles may be used to find the square roots of certain expressions which are in the form of the sum of a rational expression and a quadratic surd.

Ex. Find the square root of $13 - \sqrt{160}$.

Assume, $\sqrt{13 - \sqrt{160}} = \sqrt{x} - \sqrt{y}$. (1)

Then, by § 269, $\sqrt{13 + \sqrt{160}} = \sqrt{x} + \sqrt{y}$. (2)

Multiply (1) by (2), $\sqrt{169 - 160} = x - y$. (§ 98)

Or, $x - y = 3$. (3)

Squaring (1), $13 - \sqrt{160} = x - 2\sqrt{xy} + y$.

Whence, by § 268, $x + y = 13$. (4)

Adding (3) and (4), $2x = 16$, or $x = 8$.

Subtracting (3) from (4), $2y = 10$, or $y = 5$.

Substitute in (1), $\sqrt{13 - \sqrt{160}} = \sqrt{8} - \sqrt{5} = 2\sqrt{2} - \sqrt{5}$.

271. Examples like that of § 270 may be solved by inspection, by putting the given expression into the form of a trinomial perfect square (§ 111), as follows:

Reduce the surd term so that its coefficient may be 2.

Separate the rational term into two parts whose product shall be the expression under the radical sign of the surd term.

Extract the square root of each part, and connect the results by the sign of the surd term (§ 112).

1. Extract the square root of $8 + \sqrt{48}$.

We have, $\sqrt{48} = 2\sqrt{12}$.

We then separate 8 into two parts whose product is 12.
The parts are 6 and 2; whence,

$$\sqrt{8+\sqrt{48}}=\sqrt{6+2\sqrt{12}+2}=\sqrt{6}+\sqrt{2}.$$

2. Extract the square root of $22-3\sqrt{32}$.

We have, $3\sqrt{32}=\sqrt{9\times 8\times 4}=2\sqrt{72}$.

We then separate 22 into two parts whose product is 72.
The parts are 18 and 4; whence,

$$\sqrt{22-3\sqrt{32}}=\sqrt{18-2\sqrt{72}+4}=\sqrt{18}-\sqrt{4}=3\sqrt{2}-2.$$

EXERCISE 117

Find the square roots of the following:

1. $15+2\sqrt{54}$.
2. $21-2\sqrt{80}$.
3. $53-2\sqrt{52}$.
4. $23+6\sqrt{10}$.
5. $38-10\sqrt{13}$.
6. $29+2\sqrt{54}$.
7. $30-\sqrt{500}$.
8. $13+\sqrt{168}$.
9. $24+2\sqrt{140}$.
10. $44-4\sqrt{72}$.
11. $55-20\sqrt{6}$.
12. $55+3\sqrt{24}$.
13. $45-5\sqrt{80}$.
14. $34+12\sqrt{8}$.
15. $61+28\sqrt{3}$.
16. $53-\sqrt{600}$.
17. $60-5\sqrt{108}$.
18. $54+3\sqrt{128}$.
19. $4a-2\sqrt{4a^2-9}$.
20. $4(2x-y)+2\sqrt{15x^2-12xy}$.

Solution of Equations having the Unknown Numbers under Radical Signs.

272. 1. Solve the equation $\sqrt{x^2-5}-x=-1$.

Transposing $-x$, $\sqrt{x^2-5}=x-1$.

Squaring both members, $x^2-5=x^2-2x+1$.

Transposing, $2x=6$; whence, $x=3$.

(Substituting 3 for x in the given first member, and taking the positive value of the square root, the first member becomes

$$\sqrt{9-5}-3=2-3=-1;$$

which shows that the solution $x=3$ is correct.)

We then have the following rule:

Transpose the terms of the equation so that a surd term may stand alone in one member; then raise both members to a power of the same degree as the surd.

If surd terms still remain, repeat the operation.

The equation should be simplified as much as possible before performing the involution.

2. Solve the equation $\sqrt{2x-1}+\sqrt{2x+6}=7$.

Transposing $\sqrt{2x-1}$, $\quad \sqrt{2x+6}=7-\sqrt{2x-1}$.

Squaring, $\quad 2x+6=49-14\sqrt{2x-1}+2x-1$.

Transposing, $\quad 14\sqrt{2x-1}=42$, or $\sqrt{2x-1}=3$.

Squaring, $\quad 2x-1=9$; whence, $x=5$.

3. Solve the equation $\sqrt{x-2}-\sqrt{x}=\dfrac{1}{\sqrt{x-2}}$.

Clearing of fractions, $\quad x-2-\sqrt{x^2-2x}=1$.

Transposing, $\quad -\sqrt{x^2-2x}=3-x$.

Squaring, $\quad x^2-2x=9-6x+x^2$.

Transposing, $\quad 4x=9$, and $x=\dfrac{9}{4}$.

(If we put $x=\dfrac{9}{4}$, the given equation becomes

$$\sqrt{\frac{1}{4}}-\sqrt{\frac{9}{4}}=\frac{1}{\sqrt{\frac{1}{4}}} \tag{1}$$

If we take the *positive* value of each square root, the above is not a true equation.

But a square root may be taken as either positive or negative; and if we take the *negative* value of $\sqrt{\frac{1}{4}}$, and the *positive* value of $\sqrt{\frac{9}{4}}$, the first member of (1) becomes $-\frac{1}{2}-\frac{3}{2}$, or -2, and the second member becomes $\dfrac{1}{-\frac{1}{2}}$, or -2; then the solution $x=\frac{9}{4}$ is correct.)

4. Solve the equation $\sqrt{2-3x}+\sqrt{1+4x}=\sqrt{3+x}$.

Squaring both members,

$$2-3x+2\sqrt{2-3x}\sqrt{1+4x}+1+4x=3+x.$$

Whence, $$2\sqrt{2-3x}\sqrt{1+4x}=0;$$

or, $$\sqrt{2-3x}\sqrt{1+4x}=0.$$

Squaring, $$(2-3x)(1+4x)=0.$$

Solving as in § 126, $$2-3x=0, \text{ or } x=\frac{2}{3};$$

and $$1+4x=0, \text{ or } x=-\frac{1}{4}.$$

EXERCISE 118

Solve the following equations:

1. $\sqrt{4x+1}+5=0.$
2. $\sqrt[3]{7x-8}-2=-6.$
3. $\sqrt{16x^2+1}-4x=3.$
4. $\sqrt[3]{8x^3+36x^2}-3=2x.$
5. $\sqrt{x}+\sqrt{x+9}=-2.$
6. $\sqrt{3t-2}-\sqrt{3t}=1.$
7. $\sqrt{x+13}-\sqrt{x-5}=3.$
8. $\sqrt{5x-19}-\sqrt{5x+14}=-3.$
9. $\dfrac{2}{\sqrt{x+4}}-\dfrac{2x}{\sqrt{3-2x}}=\sqrt{3-2x}.$
10. $\dfrac{\sqrt{3x+2}+\sqrt{3x}}{\sqrt{3x+2}-\sqrt{3x}}=\dfrac{1}{4}.$
11. $\sqrt{x-5}+\sqrt{x}=\dfrac{3}{\sqrt{x}}.$
12. $\sqrt{6x}-\sqrt{6x-11}=\dfrac{5}{\sqrt{6x-11}}.$
13. $\sqrt{2s}-\sqrt{2s+5}=-\dfrac{10}{\sqrt{2s+5}}.$
14. $\sqrt{x-3}-\sqrt{x+21}=-2\sqrt{x}.$
15. $\dfrac{3\sqrt{1+2x}+4}{6\sqrt{1+2x}-1}=\dfrac{\sqrt{1+2x}+6}{2\sqrt{1+2x}-5}.$
16. $\dfrac{3\sqrt{x}+4}{5\sqrt{x}-2}=\dfrac{3\sqrt{x}+5}{5\sqrt{x}-3}.$
17. $\dfrac{\sqrt{2a-x}+\sqrt{2a+x}}{\sqrt{2a-x}-\sqrt{2a+x}}=4.$

18. $\sqrt{2n-x}-\sqrt{n-x}=3\sqrt{n}.$

19. $\sqrt{x^2+7x-4}+\sqrt{x^2-3x+1}=5.$

20. $\sqrt{x-2a}+\sqrt{x}=\dfrac{a}{\sqrt{x-2a}}.$

21. $\sqrt{x+a}+\sqrt{x+2a}=\sqrt{4x+5a}.$

22. $\sqrt{a-x}+\sqrt{b-x}=\sqrt{2a+2b}.$

23. $\sqrt{x}+\sqrt{5-2x}=\sqrt{5-x}.$

24. $\sqrt{4x-3}-\sqrt{3x-1}=\sqrt{7x-4}.$

25. $\sqrt{4p+1}-\sqrt{p-8}=\sqrt{9p-83}.$

26. $\sqrt{x-2a}-\sqrt{x-6a}=2\sqrt{x-5a}.$

27. $\sqrt{(3a+\sqrt{3ax+x^2})}=\sqrt{x}-\sqrt{3a}.$

28. $\sqrt{x}+\sqrt{4a+x}=\sqrt{4b+4x}.$

29. $\sqrt{2ax+b}+\sqrt{2ax-b}=2\sqrt{2ax-3b}.$

30. $\sqrt{a-x}+\sqrt{b-x}=\sqrt{a+b-2x}.$

31. $\sqrt{2x+5a}+\sqrt{3x+4b}=\sqrt{5x+5a+4b}.$

32. $\sqrt{2x-1}+\sqrt{3x+2}=\sqrt{3x-2}+\sqrt{2x+3}.$

33. $\sqrt{2x+5}+\sqrt{3x-2}=\sqrt{(5x+3+\sqrt{24x^2+15})}.$

IMAGINARY NUMBERS

273. It is impossible to find an even root of a negative number; for no number when raised to an even power can produce a negative result (§ 96).

An **Imaginary Number** is an indicated even root of a negative number; as $\sqrt{-2}$, or $\sqrt[4]{-3}$.

In contradistinction, rational and irrational numbers (§ 248) are called *real* numbers.

274. An imaginary number of the form $\sqrt{-a}$ is called a *pure imaginary* number, and an expression of the form $a+b\sqrt{-1}$ a *complex number.*

275. Meaning of a Pure Imaginary Number.

If $\sqrt{a}$ is *real* (§ 273), we define $\sqrt{a}$ as an expression such that, when raised to the second power, the result is a (§ 206).

To find what meaning to attach to a pure imaginary number, we assume the above principle to hold when $\sqrt{a}$ is imaginary.

Thus, $\sqrt{-2}$ means an expression such that, when raised to the second power, the result is -2; that is, $(\sqrt{-2})^2 = -2$.

In like manner, $(\sqrt{-1})^2 = -1$; etc.

OPERATIONS WITH IMAGINARY NUMBERS

276. By § 275, $\quad (\sqrt{-5})^2 = -5. \qquad (1)$

Also, $\quad (\sqrt{5}\sqrt{-1})^2 = (\sqrt{5})^2(\sqrt{-1})^2 = 5(-1) = -5. \qquad (2)$

From (1) and (2), $(\sqrt{-5})^2 = (\sqrt{5}\sqrt{-1})^2$.

Whence, $\quad \sqrt{-5} = \sqrt{5}\sqrt{-1}$.

Then, *every imaginary square root can be expressed as the product of a real number by* $\sqrt{-1}$.

$\sqrt{-1}$ is called the *imaginary unit;* it is usually represented by i.

277. Addition and Subtraction of Imaginary Numbers.

Pure imaginary numbers may be added and subtracted in the same manner as surds.

1. Add $\sqrt{-4}$ and $\sqrt{-36}$.

By § 276, $\quad \sqrt{-4} + \sqrt{-36} = 2\sqrt{-1} + 6\sqrt{-1} = 8\sqrt{-1}$.

2. Subtract $3 - \sqrt{-9}$ from $1 + \sqrt{-16}$.

In adding or subtracting complex numbers, we assume that the rules for adding or subtracting real numbers may be applied without change.

Then, $\quad 1 + \sqrt{-16} - (3 - \sqrt{-9}) = 1 + 4\sqrt{-1} - 3 + 3\sqrt{-1}$

$$= -2 + 7\sqrt{-1}.$$

EXERCISE 119

Simplify the following:

1. $\sqrt{-9} + \sqrt{-25}$. **2.** $\sqrt{-5} + \sqrt{-45}$.

3. $\sqrt{-27}-\sqrt{-12}$.

4. $\sqrt{-(x+1)^2}-\sqrt{-x^2}$.

5. $\sqrt{-a^2}+\sqrt{-b^2}-\sqrt{-c^2}$.

6. $\sqrt{-64}+\sqrt{-100}+\sqrt{-121}$.

7. $2\sqrt{-16}-5\sqrt{-49}+3\sqrt{-81}$.

8. $\sqrt{-16x^2}-\sqrt{-9x^2}-\sqrt{-4x^2}$.

9. $\sqrt{-24}-\sqrt{-54}+\sqrt{-96}$.

10. $\sqrt{-a^2-2a-1}-\sqrt{-a^2+2a-1}$.

11. Add $5+\sqrt{-4}$ to $3+\sqrt{-16}$.

12. Add $6+\sqrt{-64}$ to $1-\sqrt{-49}$.

13. Subtract $2+\sqrt{-9}$ from $8-\sqrt{-25}$.

14. Subtract $4-\sqrt{-81}$ from $7+\sqrt{-36}$.

278. Positive Integral Powers of $\sqrt{-1}$.

By § 275, $(\sqrt{-1})^2=-1$.

Then,

$$(\sqrt{-1})^3=(\sqrt{-1})^2\times\sqrt{-1}=(-1)\times\sqrt{-1}=-\sqrt{-1};$$
$$(\sqrt{-1})^4=(\sqrt{-1})^2\times(\sqrt{-1})^2=(-1)\times(-1)=1;$$
$$(\sqrt{-1})^5=(\sqrt{-1})^4\times\sqrt{-1}=1\times\sqrt{-1}=\sqrt{-1};\text{ etc.}$$

Thus, the first four positive integral powers of $\sqrt{-1}$ are $\sqrt{-1}$, -1, $-\sqrt{-1}$, and 1; and for higher powers these terms recur in the same order.

279. Multiplication of Imaginary Numbers.

The product of two or more imaginary square roots can be obtained by aid of the principles of §§ 276 and 278.

1. Multiply $\sqrt{-2}$ by $\sqrt{-3}$.

By § 276, $\sqrt{-2}\times\sqrt{-3}=\sqrt{2}\sqrt{-1}\times\sqrt{3}\sqrt{-1}$

$$=\sqrt{2}\sqrt{3}(\sqrt{-1})^2=\sqrt{6}(-1)\ (\S\ 278)=-\sqrt{6}.$$

2. Multiply together $\sqrt{-9}$, $\sqrt{-16}$, and $\sqrt{-25}$.

$$\sqrt{-9}\times\sqrt{-16}\times\sqrt{-25}=3\sqrt{-1}\times4\sqrt{-1}\times5\sqrt{-1}$$
$$=60(\sqrt{-1})^3=60(-\sqrt{-1})\ (\S\ 278)=-60\sqrt{-1}.$$

3. Multiply $2+5\sqrt{-5}$ by $4-3\sqrt{-5}$.

In multiplying complex numbers, we assume that the rules for multiplying real numbers may be applied without change.

$$\begin{array}{l} 2+\ \ 5\sqrt{-5} \\ \underline{4-\ \ 3\sqrt{-5}} \\ 8+20\sqrt{-5} \\ \underline{\quad-\ \ 6\sqrt{-5}-15(-5)} \\ 8+14\sqrt{-5}\qquad\ \ +75=83+14\sqrt{-5}. \end{array}$$

4. Expand $(\sqrt{-5}+2\sqrt{-3})^2$ by the rule of § 97.

$$(\sqrt{-5}+2\sqrt{-3})^2=(\sqrt{-5})^2+4\sqrt{5}\sqrt{-1}\times\sqrt{3}\sqrt{-1}+4(\sqrt{-3})^2$$
$$=-5+4\sqrt{15}(\sqrt{-1})^2+4(-3)=-5-4\sqrt{15}-12=-17-4\sqrt{15}.$$

EXERCISE 120

Multiply the following:

1. $\sqrt{-3}$ by $\sqrt{-5}$.
2. $\sqrt{-36}$ by $-\sqrt{-25}$.
3. $-\sqrt{-81x^2}$ by $-\sqrt{-121x^2}$.
4. $-\sqrt{-15}$ by $\sqrt{-6}$.
5. $\sqrt{-14}$ by $\sqrt{-56}$.
6. $-\sqrt{-147}$ by $-\sqrt{-45}$.
7. $5+4\sqrt{-1}$ by $2-3\sqrt{-1}$
8. $6+\sqrt{-3}$ by $7+4\sqrt{-3}$.
9. $3\sqrt{-x}-2\sqrt{-y}$ by $9\sqrt{-x}+6\sqrt{-y}$.
10. $8\sqrt{-7}-7\sqrt{-2}$ by $\sqrt{-7}-5\sqrt{-2}$.
11. $\sqrt{-a^2}$, $\sqrt{-4b^2}$, and $-\sqrt{-9c^2}$.
12. $\sqrt{-6}$, $-\sqrt{-27}$, and $-\sqrt{-54}$.
13. $\sqrt{-27}+\sqrt{-18}$ by $\sqrt{-3}-\sqrt{-2}$.
14. $2\sqrt{-3}-\sqrt{-6}$ by $\sqrt{-14}+4\sqrt{-7}$.
15. $\sqrt{-16}$, $\sqrt{-49}$, $\sqrt{-64}$, and $\sqrt{-100}$.
16. $\sqrt{-2}$, $-\sqrt{-3}$, $-\sqrt{-6}$, and $-\sqrt{-10}$.

Expand the following by the rules of §§ 97, 98:

17. $(5+\sqrt{-2})^2$.

18. $(6-\sqrt{-3})^2$.

19. $(4\sqrt{-6}+3\sqrt{-3})^2$.

20. $(3\sqrt{-5}-2\sqrt{-2})^2$.

21. $(7+2\sqrt{-1})(7-2\sqrt{-1})$.

22. $(\sqrt{-a}+b)(\sqrt{-a}-b)$.

23. $(4\sqrt{-x}+5\sqrt{-y})(4\sqrt{-x}-5\sqrt{-y})$.

24. $(8\sqrt{-2}+3\sqrt{-5})(8\sqrt{-2}-3\sqrt{-5})$.

25. $(3\sqrt{-1}+\sqrt{-3})^2+(3\sqrt{-1}-\sqrt{-3})^2$.

Expand the following by the rules of § 205:

26. $(1-\sqrt{-1})^3$.

27. $(2+\sqrt{-5})^3$.

28. Expand $(3\sqrt{-1}-\sqrt{-2}-2\sqrt{-3})^2$ by the rule of § 204.

280. Division of Imaginary Numbers.

1. Divide $\sqrt{-40}$ by $\sqrt{-5}$.

By § 276, $\dfrac{\sqrt{-40}}{\sqrt{-5}}=\dfrac{\sqrt{40}\sqrt{-1}}{\sqrt{5}\sqrt{-1}}=\dfrac{\sqrt{40}}{\sqrt{5}}=\sqrt{8}=2\sqrt{2}$.

2. Divide $\sqrt{15}$ by $\sqrt{-3}$.

$$\frac{\sqrt{15}}{\sqrt{-3}}=\frac{-\sqrt{15}\,(-1)}{\sqrt{3}\sqrt{-1}}=\frac{-\sqrt{15}\,(\sqrt{-1})^2}{\sqrt{3}\sqrt{-1}}\ (\S\ 278)=-\sqrt{5}\sqrt{-1}=-\sqrt{-5}.$$

3. Reduce $\dfrac{\sqrt{3}-\sqrt{-2}}{\sqrt{3}+\sqrt{-2}}$ to an equivalent fraction having a real denominator.

We multiply both terms of the fraction by the denominator with the sign between its terms reversed; multiplying both terms by $\sqrt{3}-\sqrt{-2}$,

$$\frac{\sqrt{3}-\sqrt{-2}}{\sqrt{3}+\sqrt{-2}}=\frac{(\sqrt{3}-\sqrt{-2})^2}{(\sqrt{3})^2-(\sqrt{-2})^2}\ (\S\ 98)$$

$$=\frac{(\sqrt{3})^2-2\sqrt{3}\sqrt{-2}+(\sqrt{-2})^2}{3-(-2)}\ (\S\ 97)$$

$$=\frac{3-2\sqrt{-6}-2}{3+2}=\frac{1-2\sqrt{-6}}{5}.$$

EXERCISE 121

Divide the following:

1. $\sqrt{-15}$ by $\sqrt{-5}$.
2. $-\sqrt{48}$ by $\sqrt{-3}$.
3. $\sqrt{-72}$ by $-\sqrt{-8}$.
4. $-\sqrt{-6xy}$ by $-\sqrt{2yz}$.
5. $\sqrt{180}$ by $-\sqrt{-10}$.
6. $-\sqrt{132}$ by $-\sqrt{-11}$.
7. $\sqrt{343}-\sqrt{-63}$ by $\sqrt{-7}$.
8. $\sqrt{-288}-\sqrt{300}$ by $-\sqrt{-6}$.

Reduce each of the following to an equivalent fraction having a real denominator:

9. $\dfrac{6}{1-\sqrt{-5}}$.
10. $\dfrac{3-\sqrt{-2}}{3+\sqrt{-2}}$.
11. $\dfrac{3\sqrt{-3}-2\sqrt{-6}}{3\sqrt{-3}+2\sqrt{-6}}$.
12. $\dfrac{2\sqrt{-5}+7\sqrt{-3}}{4\sqrt{-5}-3\sqrt{-3}}$.

281. The complex numbers $a+b\sqrt{-1}$ and $a-b\sqrt{-1}$ are called **Conjugate.**

We have $\quad (a+b\sqrt{-1})+(a-b\sqrt{-1})=2a.$

Also, $\quad (a+b\sqrt{-1})\times(a-b\sqrt{-1})$

$$=a^2-b^2(\sqrt{-1})^2=a^2+b^2 \text{ (§ 275)}.$$

Hence, *the sum and product of two conjugate complex numbers are real.*

XIX. QUADRATIC EQUATIONS

282. A **Quadratic Equation** is an equation of the second degree (§ 83), with one or more unknown numbers.

A **Pure Quadratic Equation** is a quadratic equation involving only the square of the unknown number; as, $2x^2 = 5$.

An **Affected Quadratic Equation** is a quadratic equation involving both the square and the first power of the unknown number; as, $2x^2 - 3x - 5 = 0$.

In § 126, we showed how to solve quadratic equations of the forms

$$ax^2 + bx = 0,\ ax^2 + c = 0,\ x^2 + ax + b = 0, \text{ and } ax^2 + bx + c = 0,$$

when the first members could be resolved into factors.

PURE QUADRATIC EQUATIONS

283. Let it be required to solve the equation

$$x^2 = 4.$$

Taking the square root of each member, we have

$$\pm x = \pm 2;$$

for the square root of a number may be either + or − (§ 208).

But the equations $-x = 2$ and $-x = -2$ are the same as $x = -2$ and $x = 2$, respectively, with all signs changed.

We then get all the values of x by equating the *positive* square root of the first member to ± the square root of the second.

284. A pure quadratic equation may be solved by reducing it, if necessary, to the form $x^2 = a$, and then equating x to ± the square root of a (§ 283).

1. Solve the equation $3x^2 + 7 = \frac{5x^2}{4} + 35.$

Clearing of fractions, $12x^2 + 28 = 5x^2 + 140.$

Transposing, and uniting terms, $7x^2 = 112$, or $x^2 = 16$.

Equating x to ± the square root of 16, $x = \pm 4$.

2. Solve the equation $7x^2 - 5 = 5x^2 - 13$.

Transposing, and uniting terms, $2x^2 = -8$, or $x^2 = -4$.

Equating x to $\pm$ the square root of -4, $x = \pm\sqrt{-4}$

$$= \pm 2\sqrt{-1} \text{ (§ 276).}$$

In this case, both values of x are *imaginary* (§ 274); it is impossible to find a real value of x which will satisfy the given equation.

In solving fractional quadratic equations, any solution which does not satisfy the given equation must be rejected.

Thus, let it be required to solve the equation

$$\frac{x^2 - 7}{x^2 + x - 2} = \frac{1}{x+2} - \frac{1}{x-1}.$$

Multiplying both members by $(x + 2)(x - 1)$, or $x^2 + x - 2$,

$$x^2 - 7 = x - 1 - x - 2, \text{ or } x^2 = 4.$$

Extracting square roots, $x = \pm 2$.

The solution $x = -2$ does not satisfy the given equation; the only solution is $x = 2$.

EXERCISE 122

Solve the following equations:

1. $2x^2 + 27 = 7x^2 - 53$.

2. $\frac{5}{4x^2} - \frac{13}{8x^2} = -\frac{2}{3}$.

3. $5(2x - 3) + 2x(4x + 1) = 12x - 7$.

4. $2(3x - 5)^2 + 3(x + 10)^2 = 434$.

5. $\frac{2x}{3} - \frac{5}{4x} = \frac{7x}{9} - \frac{21}{4x}$.

6. $6 - \sqrt{5x^2 - 9} = 12$.

7. $(x + a)^3 - (x - a)^3 = 8a^3$.

8. $\sqrt{x^2 - 5} = x - \frac{15}{\sqrt{x^2 - 5}}$.

9. $(2x + 7)(5x - 6) - 24x = (4x - 3)(7x + 5) - 59$.

10. $\frac{4t^2 + 3}{7} - \frac{8t^2 - 1}{2} = \frac{1}{14}$.

11. $\frac{3a}{x - 5b} - \frac{x + 5b}{3a + 10b} = 0$.

12. $\dfrac{3x^2+7}{7}-\dfrac{5x^2+3}{14}=\dfrac{4x^2-10}{35}.$

13. $(x+a)(x+b)+(x-a)(x-b)=x^2+a^2+b^2.$

14. $3\sqrt{x+1}+\sqrt{3x+7}=1.$

15. $\dfrac{10x^2-3}{18}=\dfrac{5x^2+6}{9}-\dfrac{6x^2-1}{9x^2-2}.$

16. $(k+1)(k-2)(k-3)-(k-1)(k+2)(k+3)=-52.$

17. $2x\sqrt{x^2+3}-2x\sqrt{x^2+2}=1.$

18. $\dfrac{3x^2-4}{x^2+5}-\dfrac{4x^2+3}{x^2-5}+\dfrac{2x^4+12}{x^4-25}=1.$

19. $\dfrac{x^4+3x^2-1}{2x^4-5x^2+1}=\dfrac{x^2+3}{2x^2-5}.$ **20.** $\dfrac{x^2-x+2}{x-2}-\dfrac{x^2+x-3}{x+3}=4.$

21. $\sqrt{a^2+ax+x^2}+\sqrt{a^2-ax+x^2}=a(1+\sqrt{3}).$

22. $\dfrac{1}{x+3}-\dfrac{1}{x-5}=\dfrac{x^2-17}{x^2-2x-15}.$

AFFECTED QUADRATIC EQUATIONS

285. First Method of Completing the Square.

By transposing the terms involving x to the first member, and all other terms to the second, and then dividing both members by the coefficient of x^2, any affected quadratic equation can be reduced to the form $x^2+px=q$.

We then add to both members such an expression as will make the first member a trinomial perfect square (§ 111); an operation which is termed *completing the square.*

Ex. Solve the equation $x^2+3x=4$.

A trinomial is a perfect square when its first and third terms are perfect squares and positive, and its second term plus or minus twice the product of their square roots (§ 111).

Then, the square root of the third term is equal to the second term divided by twice the square root of the first.

Hence, the *square root* of the expression which must be added to x^2+3x to make it a perfect square is $3x \div 2x$, or $\frac{3}{2}$.

Adding to both members the square of $\frac{3}{2}$, we have

$$x^2+3x+(\tfrac{3}{2})^2=4+\tfrac{9}{4}=\tfrac{25}{4}.$$

Equating the square root of the first member to $\pm$ the square root of the second (compare § 283), we have

$$x+\tfrac{3}{2}=\pm\tfrac{5}{2}.$$

Transposing $\frac{3}{2}$, $\quad x=-\frac{3}{2}+\frac{5}{2}$ or $-\frac{3}{2}-\frac{5}{2}=1$ or -4.

We then have the following rule:

Reduce the equation to the form $x^2+px=q$.

Complete the square, by adding to both members the square of one-half the coefficient of x.

Equate the square root of the first member to $\pm$ the square root of the second, and solve the linear equations thus formed.

286. 1. Solve the equation $3x^2-8x=-4$.

Dividing by 3, $\quad x^2-\dfrac{8x}{3}=-\dfrac{4}{3}$,

which is in the form $\quad x^2+px=q$.

Adding to both members the square of $\dfrac{4}{3}$, we have

$$x^2-\frac{8x}{3}+\left(\frac{4}{3}\right)^2=-\frac{4}{3}+\frac{16}{9}=\frac{4}{9}.$$

Equating the square root of the first member to $\pm$ the square root of $\dfrac{4}{9}$, $\quad x-\dfrac{4}{3}=\pm\dfrac{2}{3}$.

Transposing $-\dfrac{4}{3}$, $\quad x=\dfrac{4}{3}\pm\dfrac{2}{3}=2$ or $\dfrac{2}{3}$.

If the coefficient of x^2 is negative, the sign of each term must be changed.

2. Solve the equation $-9x^2-21x=10$.

Dividing by -9, $\quad x^2+\dfrac{7x}{3}=-\dfrac{10}{9}$.

Adding to both members the square of $\frac{7}{6}$,

$$x^2 + \frac{7x}{3} + \left(\frac{7}{6}\right)^2 = -\frac{10}{9} + \frac{49}{36} = \frac{9}{36}.$$

Extracting square roots, $\quad x + \frac{7}{6} = \pm\frac{3}{6}.$

Then, $\quad x = -\frac{7}{6} \pm \frac{3}{6} = -\frac{2}{3} \text{ or } -\frac{5}{3}.$

EXERCISE 123

Solve the following equations:

1. $x^2 + 8x = 9.$
2. $x^2 - 4x = 60.$
3. $x^2 - 9x = -14.$
4. $x^2 + 11x = -24.$
5. $x^2 - x = 20.$
6. $3x^2 - 10x = -3.$
7. $5x^2 + 3x = 8.$
8. $3x^2 + 8x = -4.$
9. $2x^2 + 9x = -4.$
10. $4x^2 + 15x - 25 = 0.$
11. $9x^2 + 14 = 27x.$
12. $18 = 4x^2 + 21x.$
13. $12 - 7x - 10x^2 = 0.$
14. $19g = -12g^2 - 5.$

287. If the coefficient of x^2 is a perfect square, it is convenient to complete the square directly by the principle stated in § 285; that is, *by adding to both members the square of the quotient obtained by dividing the coefficient of x by twice the square root of the coefficient of x^2.*

1. Solve the equation $9x^2 - 5x = 4.$

Adding to both members the square of $\frac{5}{2 \times 3}$, or $\frac{5}{6}$,

$$9x^2 - 5x + \left(\frac{5}{6}\right)^2 = 4 + \frac{25}{36} = \frac{169}{36}.$$

Extracting square roots, $3x - \frac{5}{6} = \pm\frac{13}{6}.$

Then, $\quad 3x = \frac{5}{6} \pm \frac{13}{6} = 3 \text{ or } -\frac{4}{3}, \text{ and } x = 1 \text{ or } -\frac{4}{9}.$

If the coefficient of x^2 is not a perfect square, it may be made so by multiplication.

2. Solve the equation $8x^2 - 15x = 2$.

Multiplying each term by 2, $16x^2 - 30x = 4$.

Adding to both members the square of $\frac{30}{2 \times 4}$, or $\frac{15}{4}$,

$$16x^2 - 30x + \left(\frac{15}{4}\right)^2 = 4 + \frac{225}{16} = \frac{289}{16}.$$

Extracting square roots, $4x - \frac{15}{4} = \pm\frac{17}{4}$.

Then, $4x = \frac{15}{4} \pm \frac{17}{4} = 8$ or $-\frac{1}{2}$, and $x = 2$ or $-\frac{1}{4}$.

If the coefficient of x^2 is negative, the sign of each term must be changed.

EXERCISE 124

Solve the following equations:

1. $4x^2 - 7x = -3$.
2. $9x^2 + 22x = -8$.
3. $16x^2 - 8x = 35$.
4. $8x^2 + 10x = 3$.
5. $3x^2 - 8x = 3$.
6. $18x^2 - 5x = 2$.
7. $25x^2 + 15x = 4$.
8. $36x^2 - 36x = 7$.
9. $12x^2 + x = 1$.
10. $49h^2 + 49h + 10 = 0$.
11. $64x^2 + 15 = 64x$.
12. $12 = 23e - 5e^2$.
13. $28x - 32x^2 - 3 = 0$.
14. $25x = -50x^2 - 2$.

288. Second Method of Completing the Square.

Every affected quadratic equation can be reduced to the form $ax^2 + bx + c = 0$, or $ax^2 + bx = -c$.

Multiplying both members by $4a$, we have

$$4a^2x^2 + 4abx = -4ac.$$

We complete the square by adding to both members the square of $\frac{4ab}{2 \times 2a}$ (§ 287), or b.

Then, $4a^2x^2 + 4abx + b^2 = b^2 - 4ac.$

Extracting square roots, $2ax + b = \pm\sqrt{b^2 - 4ac}.$

Transposing, $2ax = -b \pm \sqrt{b^2 - 4ac}.$

Whence, $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

We then have the following rule:

Reduce the equation to the form $ax^2 + bx = -c.$

Multiply both members by four times the coefficient of x^2, *and add to each the square of the coefficient of* x *in the given equation.*

The advantage of this method over the preceding is in avoiding fractions in completing the square.

1. Solve the equation $2x^2 - 7x = -3.$

Multiplying both members by 4×2, or 8,

$$16x^2 - 56x = -24.$$

Adding to both members the square of 7,

$$16x^2 - 56x + 7^2 = -24 + 49 = 25.$$

Extracting square roots, $4x - 7 = \pm 5.$

Then, $4x = 7 \pm 5 = 12$ or 2, and $x = 3$ or $\frac{1}{2}$.

If the coefficient of x in the given equation is *even*, fractions may be avoided, and the rule modified, as follows:

Multiply both members by the coefficient of x^2, *and add to each the square of half the coefficient of* x *in the given equation.*

2. Solve the equation $15x^2 + 28x = 32.$

Multiplying both members by 15, and adding to each the square of 14,

$$15^2x^2 + 15(28x) + 14^2 = 480 + 196 = 676.$$

Extracting square roots, $15x + 14 = \pm 26.$

Then, $15x = -14 \pm 26 = 12$ or -40, and $x = \frac{4}{5}$ or $-\frac{8}{3}$.

The method of completing the square exemplified in the present section is called the *Hindoo Method.*

EXERCISE 125

Solve the following equations:

1. $x^2+7x=18$.
2. $3x^2-2x=40$.
3. $4x^2-3x=10$.
4. $4x^2-8x=45$.
5. $8x^2+2x=3$.
6. $9x^2+18x=-8$.
7. $9x^2+4x=5$.
8. $7q^2+20q=-12$.
9. $12x^2-11x=-2$.
10. $6x^2-13x=-6$.
11. $2r^2-15r+25=0$.
12. $15x^2+26x+7=0$.
13. $5x^2+48=-32x$.
14. $13x=10x^2-3$.
15. $3=6x^2+17x$.
16. $27x-9-8x^2=0$.

289. Solution of Affected Quadratic Equations by Formula.

It follows from § 288 that, if $ax^2+bx+c=0$,

then
$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}. \qquad (1)$$

This result may be used as a *formula* for the solution of any affected quadratic equation in the form $ax^2+bx+c=0$.

1. Solve the equation $2x^2+5x-18=0$.

Here, $a=2$, $b=5$, and $c=-18$; substituting in (1),

$$x=\frac{-5\pm\sqrt{25+144}}{4}=\frac{-5\pm 13}{4}=2 \text{ or } -\frac{9}{2}.$$

2. Solve the equation $-5x^2+14x+3=0$.

Here, $a=-5$, $b=14$, $c=3$; substituting in (1),

$$x=\frac{-14\pm\sqrt{196+60}}{-10}=\frac{-14\pm 16}{-10}=-\frac{1}{5} \text{ or } 3.$$

3. Solve the equation $110x^2-21x=-1$.

Here, $a=110$, $b=-21$, $c=1$; then,

$$x=\frac{21\pm\sqrt{441-440}}{220}=\frac{21\pm 1}{220}=\frac{1}{10} \text{ or } \frac{1}{11}.$$

Particular attention must be paid to the *signs* of the coefficients in making the substitution.

EXERCISE 126

Solve the following by formula:

1. $x^2-12x+32=0.$
2. $x^2+7x-30=0.$
3. $2x^2-3x-20=0.$
4. $3x^2-x-4=0.$
5. $4x^2-5x-21=0.$
6. $20x^2+x-1=0.$
7. $9x^2-18x+8=0.$
8. $40-17x-5x^2=0.$
9. $36p^2+36p=-5.$
10. $30x^2+1=-17x.$
11. $-19x=8x^2+6.$
12. $15+22x-48x^2=0.$
13. $15x^2+26x=-8.$
14. $37x=6x^2+6.$

EXERCISE 127

The following miscellaneous equations may be solved by either of the preceding methods, preference being given to the one best adapted to the example under consideration.

In solving any fractional equation, we reject any solution which does not satisfy the given equation. (Compare last example, § 284.)

1. $\frac{x^2}{2}-\frac{7x}{6}=\frac{10}{3}.$
2. $\frac{7}{6x^2}-\frac{1}{2}=\frac{5}{12x}.$
3. $\frac{5}{6x}-\frac{13}{9x^2}=\frac{1}{18}.$
4. $\frac{3}{4t}+\frac{4t}{3}=-\frac{13}{6}.$
5. $(3x+2)(2x-3)=(4x-1)^2-14.$
6. $x(5x+22)+35=(2x+5)^2.$
7. $(x+4)(2x-1)+(2x-1)(3x+2)=(3x+2)(4x-1)-49.$
8. $\frac{24}{x-2}-\frac{24}{x}=1.$
9. $\frac{d+3}{d-2}-\frac{d+4}{d}=\frac{3}{2}.$
10. $\frac{5}{5-x}+\frac{8}{8-x}=3.$
11. $\frac{6x+5}{4x-3}=\frac{4x+4}{x-3}.$
12. $(x+1)(x+3)=12+(x+7)\sqrt{2}.$
13. $\sqrt{5x^2-3x-41}=3x-7.$
14. $(x-4)^3-(x+3)^3=-217.$
15. $\frac{3x}{4-5x}-\frac{4-5x}{3x}=-\frac{5}{6}.$

16. $\sqrt{5x+11}=\sqrt{3x+1}+2.$

17. $\sqrt{7s+8}-\sqrt{5s-4}=2.$

18. $\dfrac{x-2}{x+5}-\dfrac{x+4}{x-3}=-\dfrac{7}{3}.$

19. $\sqrt[3]{8x^3-35x^2+55x-57}=2x-3.$

20. $3\sqrt{x-1}-\dfrac{4}{\sqrt{x-1}}=4.$

22. $\dfrac{28(3x+10)}{8x^3-27}-\dfrac{25}{x(2x-3)}=0.$

21. $\dfrac{2x+1}{3x-2}+\dfrac{3x-2}{2x+1}=\dfrac{17}{4}.$

23. $\dfrac{1}{x^2-3x}-\dfrac{1}{x^2+4x}=\dfrac{14}{15x^2}.$

24. $\dfrac{1}{x-2}+\dfrac{7x}{24(x+2)}=\dfrac{15}{x^2-4}.$

25. $\dfrac{5}{2v+3}+\dfrac{7}{3v-4}=\dfrac{8v^2-13v-64}{6v^2+v-12}.$

26. $\dfrac{1}{3}\left(\dfrac{1}{4x-1}-\dfrac{1}{2}\right)=3\left(\dfrac{1}{3x+1}-\dfrac{1}{3}\right).$

27. $2\sqrt{3x+4}+3\sqrt{3x+7}=\dfrac{8}{\sqrt{3x+4}}.$

28. $\dfrac{2x^2-4x-3}{2x^2-2x+3}=\dfrac{x^2-4x+2}{x^2-3x+2}.$

29. $\dfrac{1}{x^2-4}-\dfrac{1}{3(x+2)}=1+\dfrac{3}{2-x}.$

30. $\sqrt{\dfrac{x+4}{x+5}}+\sqrt{\dfrac{x+5}{x+4}}=\dfrac{5}{2}.$

31. $\sqrt{2x}+2\sqrt{2x+5}=2\sqrt{6x+4}.$

32. $\sqrt{8x+7}=\sqrt{4x+3}+\sqrt{2x+2}.$

33. $\sqrt{2x^2+7x+7}=6-\sqrt{2x^2-9x-1}.$

34. $\sqrt{6-5x}+\sqrt{2-7x}=\sqrt{12+6x}.$

35. $\dfrac{x+1}{x-1}+\dfrac{x+2}{x-2}+\dfrac{x+3}{x-3}=3.$

(Compare Ex. 1, § 157.)

36. $\dfrac{x-2}{x-4}-\dfrac{x+2}{x+3}-\dfrac{x-2}{x-6}=-1.$

37. $\frac{x}{x-1}+\frac{x-1}{x}=\frac{x^2+x-1}{x^2-x}.$

38. $\frac{x}{x+2}-\frac{x}{x+3}=\frac{x^2+2x-2}{x^2+5x+6}.$

39. $\frac{x+4}{x}-\frac{x}{x+4}+\frac{x-5}{x}-\frac{x}{x-5}=0.$

(First combine the first two fractions, and then the last two.)

290. Solution of Literal Affected Quadratic Equations.

For the solution of literal affected quadratic equations, the methods of § 288 are usually most convenient.

1. Solve the equation $x^2+ax-bx-ab=0.$

We may write the equation $x^2+(a-b)x=ab.$

Multiplying both members by 4 times the coefficient of x^2,

$$4x^2+4(a-b)x=4ab.$$

Adding to both members the square of $a-b$,

$$4x^2+4(a-b)x+(a-b)^2=4ab+a^2-2ab+b^2$$
$$=a^2+2ab+b^2.$$

Extracting square root, $2x+(a-b)=\pm(a+b).$

Or, $\quad 2x=-(a-b)\pm(a+b).$

Then, $\quad 2x=-a+b+a+b=2b,$

or $\quad 2x=-a+b-a-b=-2a.$

Whence, $\quad x=b \text{ or } -a.$

If several terms contain the same power of x, the coefficient of that power should be enclosed in parentheses, as shown in Ex. 1.

The above equation can be solved more easily by the method of § 126; thus, by § 108, the equation may be written

$$(x+a)(x-b)=0.$$

Then, $\quad x+a=0, \text{ or } x=-a;$

and $\quad x-b=0, \text{ or } x=b.$

Several equations in Exercise 128 may be solved most easily by the method of § 126.

2. Solve the equation $(m-1)x^2 - 2m^2x = -4m^2$.

Multiplying both members by $m-1$, and adding to both the square of m^2,

$$(m-1)^2x^2 - 2m^2(m-1)x + m^4 = -4m^2(m-1) + m^4$$
$$= m^4 - 4m^3 + 4m^2.$$

Extracting square root, $(m-1)x - m^2 = \pm(m^2 - 2m)$.

Then, $\quad (m-1)x = m^2 + m^2 - 2m$ or $m^2 - m^2 + 2m$

$$= 2m(m-1) \text{ or } 2m.$$

Whence, $\quad x = 2m$ or $\dfrac{2m}{m-1}$.

In solving any fractional equation, we reject any solution which does not satisfy the given equation.

EXERCISE 128

Solve the following equations:

1. $x^2 + 2mx = 1 - m^2$.
2. $x^2 - 2ax = -6a + 9$.
3. $x^2 + (a-b)x = ab$.
4. $x^2 + nx + x = -n$.
5. $x^2 - m^2nx + mn^2x = m^3n^3$.
6. $x^2 - 4ax - 10x = -40a$.
7. $6x^2 + 4ax - 15bx = 10ab$.
8. $amx^2 - anx - bmx + bn = 0$.
9. $\sqrt{a+x} - \sqrt{2x} = \dfrac{2a}{\sqrt{a+x}}$.
10. $\dfrac{a}{2x+a} - \dfrac{a}{3x-4a} = \dfrac{4}{3}$.
11. $(a+x)^3 + (b-x)^3 = (a+b)^3$.
12. $\sqrt{(a+2b)x - 2ab} = x - 4b$.
13. $(a^2 - a - 2)x^2 - (5a-1)x = -6$.
14. $x^2 - (m-p)x + (m-n)(n-p) = 0$.
15. $(a+b)x^2 + (3a+b)x = -2a$.
16. $(b+c)x^2 - (a+c)x = b - a$.

17. $\sqrt{x+a}+2\sqrt{x+6a}=\dfrac{16a}{\sqrt{x+a}}.$

18. $\sqrt{x-a}+\sqrt{2x+3a}=\sqrt{5a}.$

19. $\dfrac{x}{a+b}+\dfrac{a+b}{x}=\dfrac{2(a^2+b^2)}{a^2-b^2}.$

20. $\dfrac{1}{a-b-x}=\dfrac{1}{a}-\dfrac{1}{b}-\dfrac{1}{x}.$

21. $\dfrac{a}{x+a-c}+\dfrac{b}{x+b-c}=2.$

22. $\dfrac{2x-3n}{3x+n}+\dfrac{3x+n}{2x-3n}=\dfrac{10}{3}.$

23. $a^2c^2(1+x)^2-b^2d^2(1-x)^2=0.$

24. $\dfrac{x^2-1}{x}=\dfrac{4ab}{a^2-b^2}.$

25. $\dfrac{2x+1}{\sqrt{x+1}}=\dfrac{2n+1}{\sqrt{n+1}}.$

26. $\sqrt{mx}+\sqrt{(m-n)x+mn}=2m.$

27. $\dfrac{x-a}{x+a}-\dfrac{x+a}{x-a}=\dfrac{x^2-5a^2}{x^2-a^2}.$

28. $\dfrac{x+a}{2x-3a}-\dfrac{3x-2a}{3x+a}=\dfrac{21ax-4a^2}{6x^2-7ax-3a^2}.$

29. $(a-b+2c)x^2-(2a+b+c)x=-a-2b+c.$

30. $\dfrac{1}{x+a}+\dfrac{1}{a}+\dfrac{1}{x+b}+\dfrac{1}{b}=0.$

31. $\dfrac{1}{x}-\dfrac{1}{b-x}+\dfrac{1}{a}+\dfrac{1}{a+b}=0.$

PROBLEMS IN PHYSICS

All the following equations occur in the study of physics.

Solve in the first six equations for the number which appears to the second power.

1. $S=\frac{1}{2}gt^2.$

2. $E=\frac{1}{2}mv^2.$

3. $F=\dfrac{mM}{d^2}.$

4. $H=C^2Rt.$

5. $f=\dfrac{mv^2}{R}.$

6. $R=\dfrac{kl}{D^2}.$

7. Solve the following equation for g; $t = \pi\sqrt{\frac{l}{g}}$.

8. Solve the following equation for t; $S = V_0 t + \frac{1}{2}gt^2$.

9. Solve the following equation for s; $V = \sqrt{2gs}$.

10. In problem 1 solve for g; in problem 7 solve for l; in problem 2 solve for m; in problem 4 solve for R; in problem 6 solve for l.

PROBLEMS INVOLVING QUADRATIC EQUATIONS WITH ONE UNKNOWN NUMBER

291. In solving problems which involve quadratic equations, there will usually be two values of the unknown number; only those values should be retained which satisfy the conditions of the problem.

1. A man sold a watch for $21, and lost as many per cent as the watch cost dollars. Find the cost of the watch.

Let $x =$ number of dollars the watch cost.

Then, $x =$ the per cent of loss,

and $x \times \frac{x}{100}$, or $\frac{x^2}{100} =$ number of dollars lost.

By the conditions, $\frac{x^2}{100} = x - 21.$

Solving, $x = 30$ or 70.

Then, the cost of the watch was either $30 or $70; for either of these answers satisfies the conditions of the problem.

2. A farmer bought some sheep for $72. If he had bought 6 more for the same money, they would have cost him $1 apiece less. How many did he buy?

Let $n =$ number bought.

Then, $\frac{72}{n} =$ number of dollars paid for one,

and $\frac{72}{n+6} =$ number of dollars paid for one if there had been 6 more.

By the conditions, $$\frac{72}{n} = \frac{72}{n+6} + 1.$$

Solving, $$n = 18 \text{ or } -24.$$

Only the *positive* value is admissible, for the negative value does not satisfy the conditions of the problem.

Therefore, the number of sheep was 18.

If, in the enunciation of the problem, the words "6 more" had been changed to "6 fewer," and "\$1 apiece less" to "\$1 apiece more," we should have found the answer 24.

3. If 3 times the square of the number of trees in an orchard be increased by 14, the result equals 23 times the number; find the number.

Let $$x = \text{number of trees.}$$

By the conditions, $$3x^2 + 14 = 23x.$$

Solving, $$x = 7 \text{ or } \tfrac{2}{3}.$$

Only the first value of x is admissible, for the fractional value does not satisfy the conditions of the problem.

Then, the number of trees is 7.

4. If the square of the number of dollars in a man's assets equals 5 times the number increased by 150, find the number.

Let $$x = \text{number of dollars in his assets.}$$

By the conditions, $$x^2 = 5x + 150.$$

Solving, $$x = 15 \text{ or } -10.$$

This means that he has assets of \$15, or liabilities of \$10.

EXERCISE 129

1. What number added to its reciprocal gives $2\frac{1}{6}$?

2. Divide the number 24 into two parts such that twice the square of the greater shall exceed 5 times the square of the less by 45.

3. Find three consecutive numbers such that the sum of their squares shall be 434.

4. Find two numbers whose difference is 7, and the difference of their cubes 721.

5. Find five consecutive numbers such that the quotient of the first by the second, added to the quotient of the fifth by the fourth, shall equal $\frac{23}{12}$.

6. Find four consecutive numbers such that if the sum of the squares of the second and fourth be divided by the sum of the squares of the first and third, the quotient shall be $\frac{13}{10}$.

7. The area of a certain square field exceeds that of another square field by 1008 square yards, and the perimeter of the greater exceeds one-half that of the smaller by 120 yards. Find the side of each field.

8. A fast train runs 8 miles an hour faster than a slow train, and takes 3 fewer hours to travel 288 miles. Find the rates of the trains.

9. The perimeter of a rectangular field is 180 feet, and its area 1800 square feet. Find its dimensions.

10. A merchant sold goods for $22.75, and lost as many per cent as the goods cost dollars. What was the cost?

11. A merchant sold two pieces of cloth of different quality for $105, the poorer containing 28 yards. He received for the finer as many dollars a yard as there were yards in the piece; and 7 yards of the poorer sold for as much as 2 yards of the finer. Find the value of each piece.

12. A merchant sold goods for $65.25, and gained as many per cent as the goods cost dollars. What was the cost?

13. A has five-fourths as much money as B. After giving A $6, B's money is equal to A's multiplied by a fraction whose numerator is 15, and whose denominator is the number of dollars A had at first. How much had each at first?

14. A and B set out at the same time from places 247 miles apart, and travel towards each other. A's rate is 9 miles an hour; and B's rate in miles an hour is less by 3 than the number of hours at the end of which they meet. Find B's rate.

15. A man buys a certain number of shares of stock, paying for each as many dollars as he buys shares. After the price has advanced one-fifth as many dollars per share as he has shares, he sells, and gains \$980. How many shares did he buy?

16. The two digits of a number differ by 1; and if the square of the number be added to the square of the given number with its digits reversed, the sum is 585. Find the number.

17. A gives \$112, in equal amounts, to a certain number of persons. B gives the same sum, in equal amounts, to 14 more persons, and gives to each \$4 less than A. How much does A give to each person?

18. The telegraph poles along a certain road are at equal intervals. If the intervals between the poles were increased by 22 feet, there would be 8 fewer in a mile. How many are there in a mile?

19. A merchant bought a cask of wine for \$48. Having lost 4 gallons by leakage, he sells the remainder at \$2 a gallon above cost, and makes a profit of 25% on his entire outlay. How many gallons did the cask contain?

20. The men in a regiment can be arranged in a column twice as long as it is wide. If their number were less by 224, they could be arranged in a hollow square 4 deep, having in each outer side of the square as many men as there were in the length of the column. Find the number of men.

21. The denominator of a fraction exceeds twice the numerator by 2, and the difference between the fraction and its reciprocal is $\frac{55}{24}$. Find the fraction.

22. A man started to walk 3 miles, intending to arrive at a certain time. After walking a mile, he was detained 10 minutes, and was in consequence obliged to walk the rest of the way a mile an hour faster. Find his original speed.

23. A regiment, in solid square, has 24 fewer men in front than when in a hollow square 6 deep. How many men are there in the regiment?

24. A rectangular field is surrounded by a fence 160 feet long. The cost of this fence, at 96 cents a foot, was one-tenth as many dollars as there are square feet in the area of the field. Find the dimensions of the field.

25. A tank can be filled by one pipe in 4 hours less time than by another; and if the pipes are open together $1\frac{1}{2}$ hours, the tank is filled. In how many hours can each pipe alone fill it? Interpret the negative answer.

26. A crew can row down stream 18 miles, and back again, in $7\frac{1}{2}$ hours. Their rate up stream is $1\frac{1}{2}$ miles an hour less than the rate of the stream. Find the rate of the stream, and of the crew in still water.

27. A man put \$5000 into a savings-bank paying a certain rate of interest. At the end of a year, he withdrew \$375, leaving the remainder at interest. At the end of another year, the amount due him was \$4968. Find the rate of interest.

28. A square garden has a square plot of grass at the centre, surrounded by a path 4 feet in width. The area of the garden outside the path exceeds by 768 square feet the area of the path; and the side of the garden is less by 16 feet than three times the side of the plot. Find the dimensions of the garden.

29. A merchant has a cask full of wine. He draws out 6 gallons, and fills the cask with water. Again he draws out 6 gallons, and fills the cask with water. There are now 25 gallons of pure wine in the cask. How many gallons does the cask hold?

30. A and B sell a quantity of corn for \$22, A selling 10 bushels more than B. If A had sold as many bushels as B did, he would have received \$8; while if B had sold as many bushels as A did, he would have received \$15. How many bushels did each sell, and at what price?

31. Two men are employed to do a certain piece of work. The first receives $48; and the second, who works 6 fewer days, receives $27. If the second had worked all the time, and the first 6 fewer days, they would have received equal amounts. How many days did each work, and at what wages?

32. A carriage-wheel, 15 feet in circumference, revolves in a certain number of seconds. If it revolved in a time longer by one second, the carriage would travel 14400 fewer feet in an hour. In how many seconds does it revolve?

PROBLEMS IN PHYSICS

1. When a body falls from rest from any point above the earth's surface, the distance, S, which it traverses in any number of seconds, t, is found to be given by the equation

$$S = \tfrac{1}{2} gt^2,$$

in which g represents the velocity which the body acquires in one second. The value of g is 32.15 feet, or 980 centimeters.

A stone fell from a balloon a mile high; how much time elapsed before it reached the earth?

2. If a body is thrown downward with an initial velocity, v_0, then the space it passes over in t seconds is found to be given by the equation

$$S = v_0t + \tfrac{1}{2} gt^2.$$

If the stone mentioned in Problem 1 had been thrown down from the balloon with a velocity of 40 feet per second, how many seconds would have elapsed before it reached the earth?

3. In the equation $t = \pi\sqrt{\dfrac{l}{g}}$, t represents the time required by a pendulum to make one vibration, l represents the length of the pendulum, and g is the same as in Problem 1. Find the length of a pendulum which beats seconds.

4. If a pendulum which beats seconds is found to be 99.3 centimeters long, find from the above equation the value of g.

5. In the equation $F = \dfrac{mM}{d^2}$, M and m represent the masses of any two attracting bodies, as, for instance, the earth and the moon, d represents the distance between these bodies, and F the force with which they attract each other.

If the moon had twice its present mass and were twice as far from the earth as at present, how much greater or less would the force of the earth's attraction be upon it than at present?

6. In the equation $E = \frac{1}{2} mv^2$, E represents the energy of a moving body, the mass of which is m and the velocity is v. Compare the energies of two bodies, one of which has twice the mass and twice the velocity of the other.

7. When a bullet is shot upward with a velocity, v, the height, S, to which it rises is given by the equation

$$v = \sqrt{2gS}.$$

Find with what velocity a body must be thrown upward to rise to the height of the Washington Monument (555 feet). (See Problem 1.)

XX. EQUATIONS SOLVED LIKE QUADRATICS

292. Equations in the Quadratic Form.

An equation is said to be in the *quadratic form* when it is expressed in three terms, two of which contain the unknown number, and *the exponent of the unknown number in one of these terms is twice its exponent in the other;* as,

$$x^6 - 6x^3 = 16;\ x^3 + x^{\frac{3}{2}} - 72 = 0;\ \text{etc.}$$

293. Equations in the quadratic form may be readily solved by the rules for quadratics.

1. Solve the equation $x^6 - 6x^3 = 16$.

Completing the square by the rule of § 285,

$$x^6 - 6x^3 + 9 = 16 + 9 = 25.$$

Extracting square roots, $x^3 - 3 = \pm 5$.

Then, $x^3 = 3 \pm 5 = 8$ or -2.

Extracting cube roots, $x = 2$ or $-\sqrt[3]{2}$.

There are also four imaginary roots, which may be found by the method of § 301.

2. Solve the equation $2x + 3\sqrt{x} = 27$.

Since $\sqrt{x}$ is the same as $x^{\frac{1}{2}}$, this is in the quadratic form.

Multiplying by 8, and adding 3^2 to both members (§ 288),

$$16x + 24\sqrt{x} + 9 = 216 + 9 = 225.$$

Extracting square roots, $4\sqrt{x} + 3 = \pm 15$.

Then, $4\sqrt{x} = -3 \pm 15 = 12$ or -18.

Whence, $\sqrt{x} = 3$ or $-\frac{9}{2}$, and $x = 9$ or $\frac{81}{4}$.

3. Solve the equation $16x^{-\frac{3}{2}} - 22x^{-\frac{3}{4}} = 3$.

Multiplying by 16, and adding 11^2 to both members,

$$16^2x^{-\frac{3}{2}} - 16 \times 22\,x^{-\frac{3}{4}} + 11^2 = 48 + 121 = 169.$$

Extracting square roots, $16\,x^{-\frac{3}{4}} - 11 = \pm 13.$

Then, $16\,x^{-\frac{3}{4}} = 11 \pm 13 = 24$ or -2, and $x^{-\frac{3}{4}} = \frac{3}{2}$ or $-\frac{1}{8}.$

Extracting cube roots, $x^{-\frac{1}{4}} = \left(\frac{3}{2}\right)^{\frac{1}{3}}$ or $-\frac{1}{2}.$

Raising to the fourth power, $x^{-1} = \left(\frac{3}{2}\right)^{\frac{4}{3}}$ or $\frac{1}{16}.$

Then, $\frac{1}{x} = \left(\frac{3}{2}\right)^{\frac{4}{3}}$ or $\frac{1}{16}$, and $x = \left(\frac{2}{3}\right)^{\frac{4}{3}}$ or $16.$

To solve an equation of the form $x^{\frac{p}{q}} = a$, first extract the root corresponding to the numerator of the fractional exponent, and afterwards raise to the power corresponding to the denominator; careful attention must be given to algebraic signs; see §§ 96 and 209.

EXERCISE 130

Solve the following equations:

1. $x^4 - 29\,x^2 = -100.$
2. $x^{-6} + 19\,x^{-3} = 216.$
3. $x^{\frac{4}{3}} - 10\,x^{\frac{2}{3}} + 9 = 0.$
4. $x^{\frac{5}{2}} + 33\,x^{\frac{5}{4}} = -32.$
5. $x^{-3} - 63\,x^{-\frac{3}{2}} - 64 = 0.$
6. $3\,x^{-2} + 14\,x^{-1} = 5.$
7. $5\,x^{-\frac{6}{5}} + 7\,x^{-\frac{3}{5}} = -2.$
8. $4\sqrt[5]{x^4} + 6 = 11\sqrt[5]{x^2}.$
9. $(3\,x^2 + 2)(2\,x^2 - 3) - 18\,x^2 = (x^2 + 3)(2\,x^2 - 4).$
10. $9(x^{-3} + 1)^2 = (x^{-3} - 4)^2 + 11\,x^{-3} - 5.$
11. $6\,h - 2 = 11\sqrt{h}.$
12. $x^{-\frac{10}{3}} + 244\,x^{-\frac{5}{3}} = -243.$
13. $3\,x^{\frac{3n}{2}} - 4\,x^{\frac{3n}{4}} = 15.$
14. $2\,s^{-8} - 35\,s^{-4} + 48 = 0.$
15. $27\,x^3 + 46 = \frac{16}{x^3}.$
16. $16\,x^8 - 33\,x^4 - 243 = 0.$
17. $32\sqrt{x^5} - 33 = -\frac{1}{\sqrt{x^5}}.$
18. $161\,x^5 + 5 = -32\,x^{10}.$
19. $\frac{81}{x^{\frac{3}{4}}} - 308 = 64\,x^{\frac{3}{4}}.$
20. $\frac{\sqrt{x}}{\sqrt{a}} + \frac{\sqrt{a}}{\sqrt{x}} = \frac{\sqrt{a}}{\sqrt{b}} + \frac{\sqrt{b}}{\sqrt{a}}.$

21. $\sqrt{6+\sqrt{x}}+\sqrt{4-\sqrt{x}}=\dfrac{12}{\sqrt{4-\sqrt{x}}}.$

22. $\sqrt{3\sqrt{x}+1}+\sqrt{\sqrt{x}-4}=\sqrt{4\sqrt{x}+5}.$

294. An equation may sometimes be solved with reference to an *expression*, by regarding it as a single letter.

1. Solve the equation $(x-5)^3-3(x-5)^{\frac{3}{2}}=40.$

Multiplying by 4, and adding 3^2 to both members,

$$4(x-5)^3-12(x-5)^{\frac{3}{2}}+3^2=160+9=169.$$

Extracting square roots, $2(x-5)^{\frac{3}{2}}-3=\pm 13.$

Then, $2(x-5)^{\frac{3}{2}}=3\pm 13=16 \text{ or } -10.$

Whence, $(x-5)^{\frac{3}{2}}=8 \text{ or } -5.$

Extracting cube roots, $(x-5)^{\frac{1}{2}}=2 \text{ or } -\sqrt[3]{5}.$

Squaring, $x-5=4 \text{ or } \sqrt[3]{25}.$

Whence, $x=9 \text{ or } 5+\sqrt[3]{25}.$

Certain equations of the fourth degree may be solved by the rules for quadratics.

2. Solve the equation $x^4+12x^3+34x^2-12x-35=0.$

The equation may be written

$$(x^4+12x^3+36x^2)-2x^2-12x=35.$$

Or, $(x^2+6x)^2-2(x^2+6x)=35.$

Completing the square, $(x^2+6x)^2-2(x^2+6x)+1=36.$

Extracting square roots, $(x^2+6x)-1=\pm 6.$

Then, $x^2+6x=7 \text{ or } -5.$

Completing the square, $x^2+6x+9=16 \text{ or } 4.$

Extracting square roots, $x+3=\pm 4 \text{ or } \pm 2.$

Then, $x=-3\pm 4 \text{ or } -3\pm 2=1,\ -7,\ -1, \text{ or } -5.$

In solving equations like the above, the first step is to complete the square with reference to the x^4 and x^3 terms; by § 287, the third term of the square is the square of the quotient obtained by dividing the x^3 term by twice the square root of the x^4 term.

3. Solve the equation $x^2 - 6x + 5\sqrt{x^2 - 6x + 20} = 46$.

Adding 20 to both members,

$$(x^2 - 6x + 20) + 5\sqrt{x^2 - 6x + 20} = 66.$$

Completing the square,

$$(x^2 - 6x + 20) + 5\sqrt{x^2 - 6x + 20} + \frac{25}{4} = 66 + \frac{25}{4} = \frac{289}{4}.$$

Extracting square roots, $\sqrt{x^2 - 6x + 20} + \frac{5}{2} = \pm\frac{17}{2}$.

Then, $\sqrt{x^2 - 6x + 20} = 6$ or -11.

Squaring, $x^2 - 6x + 20 = 36$ or 121.

Completing the square, $x^2 - 6x + 9 = 25$ or 110.

Extracting square roots, $x - 3 = \pm 5$ or $\pm\sqrt{110}$.

Then, $x = 8, -2$, or $3 \pm \sqrt{110}$.

In solving equations of the above form, add such an expression to both members that the expression without the radical sign in the first member may be the same as that within, or some multiple of it.

4. Solve the equation $\frac{x^2 - 3}{x^2 - x} + \frac{x^2 - x}{x^2 - 3} = \frac{5}{2}$.

Representing $\frac{x^2 - 3}{x^2 - x}$ by y, the equation becomes

$$y + \frac{1}{y} = \frac{5}{2}, \text{ or } 2y^2 + 2 = 5y.$$

Solving this, $y = \frac{1}{2}$ or 2; that is, $\frac{x^2 - 3}{x^2 - x} = \frac{1}{2}$ or 2.

Taking first value, $2x^2 - 6 = x^2 - x$, or $x^2 + x = 6$.

Solving, $x = 2$ or -3.

Taking second value, $x^2 - 3 = 2x^2 - 2x$, or $-x^2 + 2x = 3$

Solving, $x = 1 \pm \sqrt{-2}$.

EXERCISE 131

Solve the following equations:

1. $(2x^2 - 3x)^2 - 8(2x^2 - 3x) = 9$.

2. $x^4+10x^3+23x^2-10x-24=0.$

3. $x^4-12x^3+14x^2+132x-135=0.$

4. $5x+12+5\sqrt{5x+12}=-4.$

5. $\dfrac{x^2-3}{2x}+\dfrac{2x}{x^2-3}=-\dfrac{17}{4}.$

6. $\sqrt{5x+1}+3\sqrt[4]{5x+1}=10.$

7. $3x^2+x+5\sqrt{3x^2+x+6}=30.$

8. $8x^2-1+6x\sqrt{8x^2-1}=-8x^2.$

9. $x^4-2ax^3-17a^2x^2+18a^3x+72a^4=0.$

10. $(7x-6)^{\frac{1}{3}}-5(7x-6)^{\frac{1}{6}}=-6.$

11. $\dfrac{d^2+2}{2d-5}-\dfrac{2d-5}{d^2+2}=\dfrac{35}{6}.$

12. $x^2+7\sqrt{x^2-4x+11}=4x-23.$

13. $\sqrt{x^2-3x-3}=x^2-3x-23.$

14. $(2x^2-3x-1)^3-7(2x^2-3x-1)^{\frac{3}{2}}=8.$

15. $3\sqrt[3]{x^2-12x}-7\sqrt[6]{x^2-12x}=-2.$

16. $k^4-18k^3+109k^2-252k+180=0.$

17. $2x^2+4x+\sqrt{x^2+2x-3}=9.$

18. $7(x^3-28)^{-\frac{6}{5}}+8(x^3-28)^{-\frac{3}{5}}=-1.$

19. $(3g+15)^{-\frac{3}{2}}-5(3g+15)^{-\frac{3}{4}}=24.$

20. $9x^4-12x^3-35x^2+26x+40=0.$

21. $\dfrac{x^2-5x+1}{x^2-2x+2}-\dfrac{x^2-2x+2}{x^2-5x+1}=-\dfrac{8}{3}.$

22. $9(x+a)^{\frac{4}{3}}-22b^2(x+a)^{\frac{2}{3}}+8b^4=0.$

23. $x^2+1+\sqrt{x^2-8x+37}=8(x+12).$

24. $25(x+1)^{-1}-15(x+1)^{-\frac{1}{2}}=-2.$

25. $\sqrt{\dfrac{x^2+3}{x}}-\sqrt{\dfrac{x}{x^2+3}}=\dfrac{3}{2}.$

XXI. THEORY OF QUADRATIC EQUATIONS

295. Number of Roots.

A quadratic equation cannot have more than two different roots.

Every quadratic equation can be reduced to the form

$$ax^2 + bx + c = 0.$$

If possible, let this have three different roots, r_1, r_2, and r_3.

Then, by § 81, $$ar_1^2 + br_1 + c = 0, \quad (1)$$

$$ar_2^2 + br_2 + c = 0, \quad (2)$$

and $$ar_3^2 + br_3 + c = 0. \quad (3)$$

Subtracting (2) from (1), $a(r_1^2 - r_2^2) + b(r_1 - r_2) = 0.$

Then, $$a(r_1 + r_2)(r_1 - r_2) + b(r_1 - r_2) = 0,$$

or, $$(r_1 - r_2)(ar_1 + ar_2 + b) = 0.$$

Then, by § 126, either $r_1 - r_2 = 0$, or $ar_1 + ar_2 + b = 0$.

But $r_1 - r_2$ cannot equal 0, for, by hypothesis, r_1 and r_2 are different.

Whence, $$ar_1 + ar_2 + b = 0. \quad (4)$$

In like manner, by subtracting (3) from (1), we have

$$ar_1 + ar_3 + b = 0. \quad (5)$$

Subtracting (5) from (4), $ar_2 - ar_3 = 0$, or $r_2 - r_3 = 0$.

But this is impossible, for, by hypothesis, r_2 and r_3 are different; hence, a quadratic equation cannot have more than two different roots.

296. Sum of Roots and Product of Roots.

Let r_1 and r_2 denote the roots of $ax^2 + bx + c = 0$.

By § 289, $r_1 = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$, and $r_2 = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$.

Adding these values, $r_1 + r_2 = \frac{-2b}{2a} = -\frac{b}{a}.$

Multiplying them together,

$$r_1 r_2 = \frac{b^2 - (b^2 - 4ac)}{4a^2} \ (\S\ 98) = \frac{4ac}{4a^2} = \frac{c}{a}.$$

Hence, *if a quadratic equation is in the form* $ax^2 + bx + c = 0$, *the sum of the roots equals minus the coefficient of* x *divided by the coefficient of* x^2, *and the product of the roots equals the independent term divided by the coefficient of* x^2.

1. Find by inspection the sum and product of the roots of

$$3x^2 - 7x - 15 = 0.$$

The sum of the roots is $\frac{7}{3}$, and their product $\frac{-15}{3}$, or -5.

2. One root of the equation $6x^2 + 31x = -35$ is $-\frac{7}{2}$; find the other.

The equation can be written $6x^2 + 31x + 35 = 0$.

Then, the sum of the roots is $-\frac{31}{6}$.

Hence, the other root is $-\frac{31}{6} - \left(-\frac{7}{2}\right)$, or $-\frac{31}{6} + \frac{7}{2}$, or $-\frac{5}{3}$.

We may also find the other root by dividing the *product* of the roots, $\frac{35}{6}$, by $-\frac{7}{2}$.

We may find the values of certain other expressions which are symmetrical in the roots of the quadratic.

3. If r_1 and r_2 are the roots of $ax^2 + bx + c = 0$, find the value of $r_1^2 + r_1 r_2 + r_2^2$.

We have, $r_1^2 + r_1 r_2 + r_2^2 = (r_1 + r_2)^2 - r_1 r_2.$

But, $r_1 + r_2 = -\frac{b}{a}$, and $r_1 r_2 = \frac{c}{a}.$

Whence, $r_1^2 + r_1 r_2 + r_2^2 = \frac{b^2}{a^2} - \frac{c}{a} = \frac{b^2 - ac}{a^2}.$

EXERCISE 132

Find by inspection the sum and product of the roots of:

1. $x^2+8x+7=0$.
2. $x^2+x-20=0$.
3. $x^2-6x+1=0$.
4. $4x^2-x-5=0$.
5. $2x-14x^2=7$.
6. $10+12x-15x^2=0$.
7. $8x^2-2=-x$.
8. $9m^2x^2+21mnx+5n^2=0$.
9. One root of $x^2+7x=98$ is 7; find the other.
10. One root of $28x^2-x-15=0$ is $-\frac{5}{7}$; find the other.
11. One root of $5x^2-17x+6=0$ is $\frac{2}{5}$; find the other.

If r_1 and r_2 are the roots of $ax^2+bx+c=0$, find the values of:

12. $\dfrac{r_1^2+r_2^2}{r_1r_2}$.
13. $\dfrac{1}{r_1}+\dfrac{1}{r_2}$.
14. $\dfrac{1}{r_1^2}+\dfrac{1}{r_2^2}$.
15. $r_1^3+r_2^3$.

297. Formation of Quadratic Equations.

By aid of the principles of § 296, a quadratic equation may be formed which shall have any required roots.

For, let r_1 and r_2 denote the roots of the equation

$$ax^2+bx+c=0, \text{ or } x^2+\frac{bx}{a}+\frac{c}{a}=0. \qquad (1)$$

Then, by § 296, $\qquad \dfrac{b}{a}=-r_1-r_2$, and $\dfrac{c}{a}=r_1r_2$.

Substituting these values in (1), we have

$$x^2-r_1x-r_2x+r_1r_2=0.$$

Or, by § 108, $\qquad (x-r_1)(x-r_2)=0.$

Therefore, to form a quadratic equation which shall have any required roots,

Subtract each of the roots from x, and place the product of the resulting expressions equal to zero.

Ex. Form the quadratic whose roots shall be 4 and $-\frac{7}{4}$.

By the rule, $$(x-4)\left(x+\frac{7}{4}\right)=0.$$

Multiplying by 4, $$(x-4)(4x+7)=0;\ \text{or, } 4x^2-9x-28=0.$$

EXERCISE 133

Form the quadratic equations whose roots shall be:

1. $5,\ 8.$
2. $-4,\ 3.$
3. $-1,\ -\frac{3}{4}.$
4. $6,\ -\frac{13}{2}.$
5. $\frac{1}{7},\ -\frac{4}{3}.$
6. $\frac{16}{9},\ 0.$
7. $-\frac{5}{8},\ -\frac{6}{5}.$
8. $-\frac{1}{6},\ \frac{3}{10}.$
9. $a+2b,\ a-2b.$
10. $3m-n,\ m+4n.$
11. $-4+5\sqrt{3},\ -4-5\sqrt{3}.$
12. $\dfrac{\sqrt{m}+2\sqrt{n}}{2},\ \dfrac{\sqrt{m}-2\sqrt{n}}{2}.$

FACTORING

298. Factoring of Quadratic Expressions.

A *quadratic expression* is an expression of the form

$$ax^2+bx+c.$$

In § 117, we showed how to factor certain expressions of this form *by inspection;* we will now derive a rule for factoring any quadratic expression; we have,

$$ax^2+bx+c=a\left(x^2+\frac{bx}{a}+\frac{c}{a}\right)$$

$$=a\left[x^2+\frac{bx}{a}+\left(\frac{b}{2a}\right)^2-\frac{b^2}{4a^2}+\frac{c}{a}\right]$$

$$=a\left[\left(x+\frac{b}{2a}\right)^2-\frac{b^2-4ac}{4a^2}\right]$$

$$=a\left(x+\frac{b}{2a}+\frac{\sqrt{b^2-4ac}}{2a}\right)\left(x+\frac{b}{2a}-\frac{\sqrt{b^2-4ac}}{2a}\right),$$

by § 114.

But by § 289, the roots of $ax^2+bx+c=0$ are

$$-\frac{b}{2a}+\frac{\sqrt{b^2-4ac}}{2a} \text{ and } -\frac{b}{2a}-\frac{\sqrt{b^2-4ac}}{2a}.$$

Hence, *to factor a quadratic expression, place it equal to zero, and solve the equation thus formed.*

Then the required factors are the coefficient of x^2 in the given expression, x minus the first root, and x minus the second.

1. Factor $6x^2+7x-3$.

Solving the equation $6x^2+7x-3=0$, by § 289,

$$x=\frac{-7\pm\sqrt{49+72}}{12}=\frac{-7\pm 11}{12}=\frac{1}{3}\text{ or }-\frac{3}{2}.$$

Then, $$6x^2+7x-3=6\left(x-\frac{1}{3}\right)\left(x+\frac{3}{2}\right)$$

$$=3\left(x-\frac{1}{3}\right)\times 2\left(x+\frac{3}{2}\right)=(3x-1)(2x+3).$$

2. Factor $4+13x-12x^2$.

Solving the equation $4+13x-12x^2=0$, by § 289,

$$x=\frac{-13\pm\sqrt{169+192}}{-24}=\frac{-13\pm 19}{-24}=-\frac{1}{4}\text{ or }\frac{4}{3}.$$

Whence, $$4+13x-12x^2=-12\left(x+\frac{1}{4}\right)\left(x-\frac{4}{3}\right)$$

$$=4\left(x+\frac{1}{4}\right)\times\left(-3\right)\left(x-\frac{4}{3}\right).$$

$$=(1+4x)(4-3x).$$

3. Factor $2x^2-3xy-2y^2-7x+4y+6$.

We solve $$2x^2-x(3y+7)-2y^2+4y+6=0.$$

By § 289, $$x=\frac{3y+7\pm\sqrt{(3y+7)^2+16y^2-32y-48}}{4}$$

$$=\frac{3y+7\pm\sqrt{25y^2+10y+1}}{4}=\frac{3y+7\pm(5y+1)}{4}$$

$$=\frac{8y+8}{4}\text{ or }\frac{-2y+6}{4}=2y+2\text{ or }\frac{-y+3}{2}.$$

Then, $$2x^2-3xy-2y^2-7x+4y+6$$

$$=2[x-(2y+2)]\left[x-\frac{-y+3}{2}\right]$$

$$=(x-2y-2)(2x+y-3).$$

EXERCISE 134

Factor the following:

1. $x^2+14x+33$.
2. $x^2-13x+40$.
3. a^2-a-42.
4. $2x^2+11x-6$.
5. $4x^2+19x+12$.
6. $8m^2-14m+5$.
7. $9-8x-x^2$.
8. $10x^2+39x+14$.
9. $40+19x-3x^2$.
10. $35-x-6x^2$.
11. $5x^2-21nx+18n^2$.
12. $7x^2-18ax+8a^2$.
13. $21x^2+5x-6$.
14. $12x^2-16x-35$.
15. $3+13x-30x^2$.
16. $28-5t-12t^2$.
17. $16x^2+30x+9$.
18. $18x^2-31x+6$.
19. $6x^2-23x-35$.
20. $20-13x-15x^2$.
21. $20x^2+33mx+7m^2$.
22. $24x^2-38xyz+15y^2z^2$.
23. $x^2-xy-6y^2-6x+13y+5$.
24. $x^2-3xy-4y^2+6x-4y+8$.
25. $x^2-6xy+5y^2-2x-2y-3$.
26. $2a^2+5ab+2b^2+7a+5b+3$.
27. $3x^2+7xy-6y^2-10x-8y+8$.
28. $2-7y-7x+3y^2+xy-4x^2$.
29. $6x^2-2xy-20y^2+5xz+23yz-6z^2$.

299. If the coefficient of x^2 is a perfect square, it is preferable to factor the expression by completing the square as in § 287, and then using § 114.

1. Factor $9x^2-9x-4$.

By § 287, $9x^2-9x$ will become a perfect square by adding to it the square of $\frac{9}{2\times 3}$, or $\frac{3}{2}$.

Then, $$9x^2-9x-4=9x^2-9x+\left(\frac{3}{2}\right)^2-\frac{9}{4}-4=\left(3x-\frac{3}{2}\right)^2-\frac{25}{4}$$
$$=\left(3x-\frac{3}{2}+\frac{5}{2}\right)\left(3x-\frac{3}{2}-\frac{5}{2}\right) \qquad (\S\ 114)$$
$$=(3x+1)(3x-4).$$

If the x^2 term is negative, the entire expression should be enclosed in parentheses preceded by a $-$ sign.

2. Factor $3-12x-4x^2$.

$$3-12x-4x^2=-(4x^2+12x-3)$$
$$=-(4x^2+12x+9-9-3)$$
$$=-[(2x+3)^2-12]$$
$$=(2x+3+\sqrt{12})\times(-1)(2x+3-\sqrt{12})$$
$$=(2\sqrt{3}+3+2x)(2\sqrt{3}-3-2x).$$

EXERCISE 135

Factor the following:

1. $4x^2-12x-7$.
2. $9x^2-21x+10$.
3. x^2+x-12.
4. $16x^2+40x+21$.
5. $9x^2+24x-2$.
6. $4x^2+20x+19$.
7. $1+2x-x^2$.
8. $16x^2-16x+1$.
9. $6-5x-25x^2$.
10. $4x^2+9x-9$.
11. $36x^2+72x+29$.
12. $25x^2-10x-11$.

300. We will now take up the factoring of expressions of the forms $x^4+ax^2y^2+y^4$, or x^4+y^4, when the factors involve surds. (Compare § 115.)

1. Factor $a^4+2a^2b^2+25b^4$.

$$a^4+2a^2b^2+25b^4=(a^4+10a^2b^2+25b^4)-8a^2b^2$$
$$=(a^2+5b^2)^2-(ab\sqrt{8})^2$$
$$=(a^2+5b^2+ab\sqrt{8})(a^2+5b^2-ab\sqrt{8})$$
$$=(a^2+2ab\sqrt{2}+5b^2)(a^2-2ab\sqrt{2}+5b^2).$$

2. Factor x^4+1.

$$\begin{aligned} x^4+1 &= (x^4+2x^2+1)-2x^2 \\ &= (x^2+1)^2-(x\sqrt{2})^2 \\ &= (x^2+x\sqrt{2}+1)(x^2-x\sqrt{2}+1). \end{aligned}$$

EXERCISE 136

In each of the following obtain two sets of factors, when this can be done without bringing in imaginary numbers:

1. x^4-7x^2+4.
2. a^4+b^4.
3. $9m^4-11m^2+1$.
4. $4a^4+6a^2+9$.
5. $36x^4-92x^2+49$.
6. $25m^4+28m^2n^2+16n^4$.

301. Solution of Equations by Factoring.

In § 126, we showed how to solve equations whose first members could be resolved by inspection into first degree factors, and whose second members were zero.

We will now take up equations whose first members can be resolved into factors partly of the first and partly of the second, or entirely of the second degree.

1. Solve the equation $x^3+1=0$.

Factoring the first member, $(x+1)(x^2-x+1)=0$.

Then, $x+1=0$, or $x=-1$;

and $x^2-x+1=0$; whence, by § 289, $x=\frac{1\pm\sqrt{1-4}}{2}=\frac{1\pm\sqrt{-3}}{2}$.

2. Solve the equation $x^4+1=0$.

By Ex. 2, § 300, $x^4+1=(x^2+x\sqrt{2}+1)(x^2-x\sqrt{2}+1)$.

Solving $x^2+x\sqrt{2}+1=0$, we have

$$x=\frac{-\sqrt{2}\pm\sqrt{2-4}}{2}=\frac{-\sqrt{2}\pm\sqrt{-2}}{2}.$$

Solving $x^2-x\sqrt{2}+1=0$, we have

$$x=\frac{\sqrt{2}\pm\sqrt{2-4}}{2}=\frac{\sqrt{2}\pm\sqrt{-2}}{2}.$$

The above examples illustrate the important principle that the degree of an equation indicates the number of its roots; thus, an equation of the third degree has three roots; of the fourth degree, four roots; etc.

The roots are not necessarily *unequal;* thus, the equation $x^2-2x+1=0$ may be written $(x-1)(x-1)=0$, and its two roots are 1 and 1.

EXERCISE 137

Solve the following equations:

1. $5x^3-3x^2-9x=0.$
2. $(x+4)(2x^2+5x+25)=0.$
3. $(9x^2-4)(11x^2+8x-4)=0.$
4. $x^4-11a^2x^2-12a^4=0.$
5. $x^4-81=0.$
6. $x^3+2x^2+2x+4=0.$
7. $x^3-1=0.$
8. $x^4+8x=0.$
9. $5x^3-4x^2+60x-48=0.$
10. $27x^3+8a^3=0.$
11. $x^4-6x^2+1=0.$
12. $x^4-5x^2+1=0.$
13. $64x^3-125=0.$
14. $x^4-10x^2+9=0.$
15. $x^4-20x^2+16=0.$
16. $9x^4+5x^2+4=0.$
17. $x^6-729=0.$
18. $x^8-256=0.$
19. $\dfrac{x^2+2x+4}{x^2-2x-4}=\dfrac{4}{x^2}.$
20. $9x^4-x^2+4=0.$
21. $\sqrt{x^2+1+\sqrt{9x^4-x}}=2x-1.$

302. Discussion of General Equation.

By § 289, the roots of $ax^2+bx+c=0$ are

$$r_1=\frac{-b+\sqrt{b^2-4ac}}{2a} \text{ and } r_2=\frac{-b-\sqrt{b^2-4ac}}{2a}.$$

We will now discuss these results for all possible real values of a, b, and c.

I. b^2-4ac *positive.*

In this case, r_1 and r_2 are *real* and *unequal.*

II. $b^2-4ac=0.$

In this case, r_1 and r_2 are *real* and *equal.*

III. $b^2 - 4ac$ *negative.*

In this case, r_1 and r_2 are *imaginary* (§ 273).

IV. $b = 0.$

In this case, the equation takes the form

$$ax^2 + c = 0; \text{ whence, } x = \pm\sqrt{-\frac{c}{a}}.$$

If a and c are of unlike sign, the roots are *real, equal in absolute value,* and *unlike in sign.*

If a and c are of like sign, both roots are *imaginary.*

V. $c = 0.$

In this case, the equation takes the form

$$ax^2 + bx = 0; \text{ whence, } x = 0 \text{ or } -\frac{b}{a}.$$

Hence, the roots are both *real,* one being *zero.*

VI. $b = 0,$ *and* $c = 0.$

In this case, the equation takes the form $ax^2 = 0.$

Hence, both roots equal zero.

The roots are both *rational,* or both *irrational,* according as $b^2 - 4ac$ is, or is not, a perfect square.

Ex. Determine by inspection the nature of the roots of

$$2x^2 - 5x - 18 = 0.$$

Here $a = 2$, $b = -5$, $c = -18$; and $b^2 - 4ac = 25 + 144 = 169.$
Since $b^2 - 4ac$ is positive, the roots are real and unequal.
Since $b^2 - 4ac$ is a perfect square, both roots are rational.

EXERCISE 138

Determine by inspection the nature of the roots of the following:

1. $6x^2 + 17x + 5 = 0.$
2. $6x^2 + x = 0.$
3. $4x^2 - 28x + 49 = 0.$
4. $12x^2 - 19x + 4 = 0.$
5. $25x^2 - 4 = 0.$
6. $x^2 - 19x + 125 = 0.$
7. $5x^2 - 4x = 0$
8. $9x^2 + 6x - 1 = 0.$
9. $16x^2 + 24x + 9 = 0.$
10. $30x^2 - 30 = 11x.$

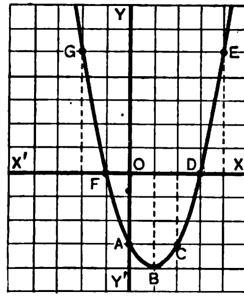
GRAPHICAL REPRESENTATION OF QUADRATIC EXPRESSIONS WITH ONE UNKNOWN NUMBER

303. The graph of a quadratic expression, with one unknown number, x, may be found by putting y equal to the expression, and finding the graph of the resulting equation as in § 181.

1. Find the graph of $x^2 - 2x - 3$.

Put $y = x^2 - 2x - 3$.

If $x = 0$,	$y = -3$.	(A)
If $x = 1$,	$y = -4$.	(B)
If $x = 2$,	$y = -3$.	(C)
If $x = 3$,	$y = 0$.	(D)
If $x = 4$,	$y = 5$.	(E)
If $x = -1$,	$y = 0$.	(F)
If $x = -2$,	$y = 5$.	(G)



Scale $\frac{1}{8}$ inch = 1 unit.

The graph is the *curve GBE*.

By taking other values for x, the curve may be traced beyond E and G; it extends in either direction to an indefinitely great distance from XX'.

[We may determine the *lowest* point of the curve by the artifice of completing the square as follows:

We have, $x^2 - 2x - 3 = (x^2 - 2x + 1) - 1 - 3 = (x-1)^2 - 4$.

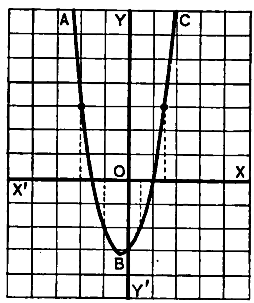
The latter expression has its negative value of greatest absolute value when $x = 1$, being then equal to -4.

Then, the lowest point of the curve has the co-ordinates $(1, -4)$; and is therefore the point B].

2. Find the graph of $2x^2 + x - 3$.

Put $y = 2x^2 + x - 3$.

If $x = 0$,	$y = -3$.
If $x = 1$,	$y = 0$.
If $x = 2$,	$y = 7$.
If $x = -1$,	$y = -2$.
If $x = -2$,	$y = 3$.



Scale: $\frac{1}{8}$ inch.

The graph is the curve ABC.

[To find the lowest point of the curve, we have

$$2x^2 + x - 3 = 2\left(x^2 + \frac{x}{2}\right) - 3 = 2\left(x^2 + \frac{x}{2} + \frac{1}{16}\right) - \frac{2}{16} - 3 = 2\left(x + \frac{1}{4}\right)^2 - \frac{25}{8}.$$

The latter expression has its negative value of greatest absolute value when $x = -\frac{1}{4}$, being then equal to $-\frac{25}{8}$.

Then, the lowest point has the co-ordinates $\left(-\frac{1}{4}, -\frac{25}{8}\right)$.]

304. The principle of § 188 holds for the graph of the first member of any quadratic equation, with one unknown number.

Thus, the graph of $x^2 - 2x - 3$ (§ 303) intersects the axis XX' at points whose abscissas are 3 and -1, and the equation $x^2 - 2x - 3 = 0$ has the roots 3 and -1.

Again, the graph of $2x^2 + x - 3$ intersects XX' at the point whose abscissa is 1, and between the points whose abscissas are -1 and -2; and the equation $2x^2 + x - 3 = 0$ has one root equal to 1, and one between -1 and -2.

EXERCISE 139

Find the graph of the first member of each of the following equations, and verify the principle of § 188 in the results:

1. $x^2 - 5x + 4 = 0$.
2. $x^2 + x - 6 = 0$.
3. $x^2 + 7x + 10 = 0$.
4. $3x^2 - 4x = 0$.
5. $4x^2 + 7x = 0$.
6. $2x^2 - 11x - 6 = 0$.
7. $6x^2 + 5x - 6 = 0$.
8. $8x^2 - 14x - 15 = 0$.

305. Graphs of the First Members of Quadratic Equations having Equal or Imaginary Roots.

1. Consider the equation $x^2 - 4x + 4 = 0$.

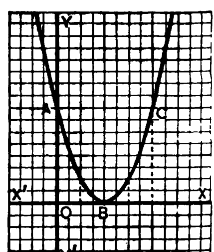
We may write the equation $(x-2)(x-2) = 0$.

Then, by § 126, the roots are 2 and 2.

To find the graph of the first member, put $y = (x-2)^2$.

If $x = 0$, $y = 4$. If $x = 2$, $y = 0$.

If $x = 1$, $y = 1$. If $x = 3$, $y = 1$; etc.



Scale ⅛ inch.

The graph is the curve ABC, which extends to an indefinitely great distance from XX'.

Since $(x-2)^2$ cannot be negative for any value of x, y cannot be negative; and the graph is *tangent* to XX'.

It is evident from this that, if a quadratic equation, with one unknown number, has equal roots, the graph of its first member is tangent to XX'.

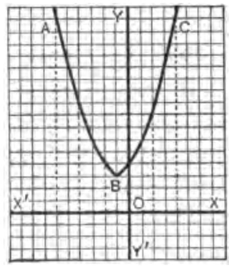
2. Consider the equation $x^2+x+2=0$.

Solving, $\quad x = \dfrac{-1 \pm \sqrt{-7}}{2}.$

To find the graph of the first member, put $y = x^2 + x + 2$.

If $x = 0$, $y = 2$. If $x = -1$, $y = 2$.

If $x - 1$, $y = 4$. If $x = -2$, $y = 4$; etc.



Scale: $\frac{1}{8}$ inch.

The graph is the curve ABC, which extends to an indefinitely great distance from XX'.

We have, $\quad x^2 + x + 2 = \left(x^2 + x + \dfrac{1}{4}\right) - \dfrac{1}{4} + 2 = \left(x + \dfrac{1}{2}\right)^2 + \dfrac{7}{4}.$

Since $\left(x + \dfrac{1}{2}\right)^2 + \dfrac{7}{4}$ cannot be zero or negative for any value of x, y cannot be zero or negative, and the graph does not intersect XX'.

It is evident from this that, if a quadratic equation, with one unknown number, has imaginary roots, the graph of its first member does not intersect XX'.

EXERCISE 140

Find the graphs of the first members of the following, and in each case verify the above principles:

1. $x^2 - 6x + 9 = 0$.
2. $x^2 + 3x + 4 = 0$.
3. $4x^2 + 4x + 1 = 0$.
4. $2x^2 - 4x + 5 = 0$.

XXII. SIMULTANEOUS QUADRATIC EQUATIONS

306. *On the use of the double signs* $\pm$ *and* $\mp$.

If two or more equations involve double signs, it will be understood that the equations can be read in two ways; first, reading all the *upper* signs together; second, reading all the *lower* signs together.

Thus, the equations $x = \pm 2$, $y = \pm 3$, can be read either

$$x = +2,\ y = +3, \text{ or } x = -2,\ y = -3.$$

Also, the equations $x = \pm 2$, $y = \mp 3$, can be read either

$$x = +2,\ y = -3, \text{ or } x = -2,\ y = +3.$$

307. Two equations of the second degree (§ 83) with two unknown numbers will generally produce, by elimination, an equation of the *fourth* degree with one unknown number.

Consider, for example, the equations

$$\begin{cases} x^2 + y = a. & (1) \\ x + y^2 = b. & (2) \end{cases}$$

From (1), $y = a - x^2$; substituting in (2),

$$x + a^2 - 2\,ax^2 + x^4 = b;$$

an equation of the fourth degree in x.

The methods already given are, therefore, not sufficient for the solution of every system of simultaneous quadratic equations, with two unknown numbers.

In certain cases, however, the solution may be effected.

308. Case I. *When each equation is in the form*

$$ax^2 + by^2 = c.$$

In this case, either x^2 or y^2 can be eliminated by addition or subtraction.

1. Solve the equations $\begin{cases} 3x^2 + 4y^2 = 76. & (1) \\ 3y^2 - 11x^2 = 4. & (2) \end{cases}$

Multiply (1) by 3, $9x^2 + 12y^2 = 228.$
Multiply (2) by 4, $12y^2 - 44x^2 = 16.$

Subtracting, $53x^2 = 212.$

Then, $x^2 = 4$, and $x = \pm 2.$

Substituting $x = \pm 2$ in (1), $12 + 4y^2 = 76$, or $4y^2 = 64.$

Then, $y^2 = 16$, and $y = \pm 4.$

The solution is $x = 2$, $y = \pm 4$; or, $x = -2$, $y = \pm 4.$

In this case there are four possible sets of values of x and y which satisfy the given equations:

1. $x = 2$, $y = 4.$
2. $x = 2$, $y = -4.$
3. $x = -2$, $y = 4.$
4. $x = -2$, $y = -4.$

It would not be correct to leave the result in the form $x = \pm 2$, $y = \pm 4$, for this represents only the first and fourth of the above sets of values.

The method of elimination by addition or subtraction may be used in other examples.

2. Solve the equations $\begin{cases} 3x^2 - 4y = 47. & (1) \\ 7x^2 + 6y = 33. & (2) \end{cases}$

Multiply (1) by 3, $9x^2 - 12y = 141.$
Multiply (2) by 2, $14x^2 + 12y = 66.$

Adding, $23x^2 = 207.$

Then, $x^2 = 9$, and $x = \pm 3.$

Substituting $x = \pm 3$ in (1), $27 - 4y = 47$, and $y = -5.$

It is possible to eliminate one unknown number, in the above examples, by *substitution* (§ 169), or by *comparison* (§ 170).

EXERCISE 141

Solve the following equations:

1. $\begin{cases} 3x^2 + 2y^2 = 66. \\ 9x^2 + 5y^2 = 189. \end{cases}$

2. $\begin{cases} 3x - 5y^2 = -116. \\ 7x + 4y^2 = 121. \end{cases}$

3. $\begin{cases} 3x^2 - 2xy = 24. \\ 4x^2 - 5xy = 46. \end{cases}$

6. $\begin{cases} 11x^2 - 6y^2 = 84. \\ 7x^2 + 15y^2 = 204. \end{cases}$

4. $\begin{cases} 4h^2 + 9k^2 = 13. \\ 8h^2 - 27k^2 = 6. \end{cases}$

7. $\begin{cases} 2x^2 - xy - 3y^2 = 0. \\ x^2 + xy + 3y^2 = 27. \end{cases}$

5. $\begin{cases} 5xy + y^2 = -75. \\ xy - 3y^2 = -95. \end{cases}$

8. $\begin{cases} 2x^2 + 3y^2 + x = 67. \\ x^2 - 2y^2 = 17. \end{cases}$

9. $\begin{cases} 4x^2 - y^2 = 3a^2 + 10ab + 3b^2. \\ 4y^2 - x^2 = 3a^2 - 10ab + 3b^2. \end{cases}$

309. Case. II. *When one equation is of the second degree, and the other of the first.*

Equations of this kind may be solved by finding one of the unknown numbers in terms of the other from the first degree equation, and substituting this value in the other equation.

Ex. Solve the equations $\begin{cases} 2x^2 - xy = 6y. & (1) \\ x + 2y = 7. & (2) \end{cases}$

From (2), $2y = 7 - x$, or $y = \frac{7-x}{2}$. (3)

Substituting in (1), $2x^2 - x\left(\frac{7-x}{2}\right) = 6\left(\frac{7-x}{2}\right)$.

Clearing of fractions, $4x^2 - 7x + x^2 = 42 - 6x$, or $5x^2 - x = 42$.

Solving, $x = 3$ or $-\frac{14}{5}$.

Substituting in (3), $y = \frac{7-3}{2}$ or $\frac{7+\frac{14}{5}}{2} = 2$ or $\frac{49}{10}$.

The solution is $x = 3$, $y = 2$; or, $x = -\frac{14}{5}$, $y = \frac{49}{10}$.

Certain examples where one equation is of the *third* degree and the other of the first may be solved by the method of Case II.

EXERCISE 142

Solve the following equations:

1. $\begin{cases} x^2 + 3y^2 = 37. \\ x - 2y = 9. \end{cases}$

2. $\begin{cases} x + y = -4. \\ xy = -45. \end{cases}$

3. $\begin{cases} x^2+xy+y^2=97. \\ x-y=5. \end{cases}$

4. $\begin{cases} x^2-xy+2y^2=8. \\ 3x+y=10. \end{cases}$

5. $\begin{cases} x^3+y^3=91. \\ x+y=7. \end{cases}$

6. $\begin{cases} x^2-xy+y^2=124. \\ x+y=8. \end{cases}$

7. $\begin{cases} 3e^2-eg-g^2=-3. \\ 2e-3g=5. \end{cases}$

8. $\begin{cases} x^3-y^3=344. \\ x-y=8. \end{cases}$

9. $\begin{cases} x-y=a+2b. \\ x^2+y^2=a^2+2ab+2b^2. \end{cases}$

10. $\begin{cases} \dfrac{2x}{3}+\dfrac{3y}{2}=2. \\ \dfrac{3}{2x}+\dfrac{2}{3y}=2. \end{cases}$

11. $\begin{cases} xy=a^2+a-2. \\ 3x+4y=7a+2. \end{cases}$

12. $\begin{cases} \dfrac{x-y}{x}-\dfrac{x}{x-y}=\dfrac{40}{21}. \\ 2y+3x=-1. \end{cases}$

13. $\begin{cases} \dfrac{p}{2t}-\dfrac{t}{3p}=\dfrac{29}{24}. \\ 4p-t=-2. \end{cases}$

14. $\begin{cases} 2x-3y=-1. \\ \dfrac{x^2-y}{32}-\dfrac{y^2-x}{15}=0. \end{cases}$

310. Case III. *When the given equations are symmetrical with respect to x and y; that is, when x and y can be interchanged without changing the equation.*

Equations of this kind may be solved by combining them in such a way as to obtain the values of $x+y$ and $x-y$.

1. Solve the equations $\begin{cases} x+y=2. & (1) \\ xy=-15. & (2) \end{cases}$

Squaring (1), $x^2+2xy+y^2=4.$

Multiplying (2) by 4, $4xy=-60.$

Subtracting, $x^2-2xy+y^2=64.$

Extracting square roots, $x-y=\pm 8.$ (3)

Adding (1) and (3), $2x=2\pm 8=10$ or $-6.$

Whence, $x=5$ or $-3.$

Subtracting (3) from (1), $2y=2\mp 8=-6$ or $10.$

Whence, $y=-3$ or $5.$

The solution is $x=5$, $y=-3$; or, $x=-3$, $y=5$.

In subtracting ± 8 from 2, we have 2 ∓ 8, in accordance with the notation explained in § 306.

In operating with double signs, $\pm$ is changed to $\mp$, and $\mp$ to $\pm$, whenever $+$ should be changed to $-$.

(The above equations may also be solved by the method of Case II; but the symmetrical method is shorter and neater.)

2. Solve the equations $\begin{cases} x^2 + y^2 = 50. & (1) \\ xy = -7. & (2) \end{cases}$

Multiply (2) by 2, $\quad 2xy = -14. \quad (3)$

Add (1) and (3), $\quad x^2 + 2xy + y^2 = 36$, or $x + y = \pm 6. \quad (4)$

Subtract (3) from (1), $\quad x^2 - 2xy + y^2 = 64$, or $x - y = \pm 8. \quad (5)$

Add (4) and (5), $\quad 2x = 6 \pm 8$, or -6 ± 8.

Whence, $\quad x = 7, -1, 1$, or -7.

Subtract (5) from (4), $\quad 2y = 6 \mp 8$, or -6 ∓ 8.

Whence, $\quad y = -1, 7, -7$, or 1.

The solution is $x = \pm 7,\ y = \mp 1$; or, $x = \pm 1,\ y = \mp 7$.

Certain examples in which one equation is of the *third* degree, and the other of the first or second, may be solved by the method of Case III.

3. Solve the equations $\begin{cases} x^3 - y^3 = 56. & (1) \\ x^2 + xy + y^2 = 28. & (2) \end{cases}$

Divide (1) by (2), $\quad x - y = 2. \quad (3)$

Squaring (3), $\quad x^2 - 2xy + y^2 = 4. \quad (4)$

Subtract (4) from (2), $\quad 3xy = 24$, or $xy = 8. \quad (5)$

Add (2) and (5), $\quad x^2 + 2xy + y^2 = 36$, or $x + y = \pm 6. \quad (6)$

Add (3) and (6), $\quad 2x = 2 \pm 6 = 8$ or -4.

Whence, $\quad x = 4$ or -2.

Subtract (3) from (6), $\quad 2y = \pm 6 - 2 = 4$ or -8.

Whence, $\quad y = 2$ or -4.

The solution is $\quad x = 4,\ y = 2$; or, $x = -2,\ y = -4$.

(If we interchange x and y in equation (1), it becomes

$$y^3 - x^3 = 56, \text{ or } \qquad x^3 - y^3 = -56,$$

which is not the same as (1).

Thus, the equation (1) is not symmetrical with respect to x and y; but the method of Case III may often be used when either or both of the given equations are symmetrical, except with respect to the *signs* of the terms.)

We may advantageously use the method of Case III to solve certain equations which are not symmetrical with respect to x and y; as, for example, the equations

$$\begin{cases} x - 2y = -4. \\ x^2 + 4y^2 = 40. \end{cases}$$

EXERCISE 143

Solve the following equations by the symmetrical method:

1. $\begin{cases} x^2 + y^2 = 29. \\ x + y = -3. \end{cases}$

2. $\begin{cases} x - y = 11. \\ xy = -28. \end{cases}$

3. $\begin{cases} q^2 + s^2 = 130. \\ q - s = -8. \end{cases}$

4. $\begin{cases} x^2 + y^2 = 40. \\ xy = 12. \end{cases}$

5. $\begin{cases} x^3 - y^3 = 35. \\ x^2 + xy + y^2 = 7. \end{cases}$

6. $\begin{cases} x^3 - y^3 = 26. \\ x - y = 2. \end{cases}$

7. $\begin{cases} x + y = 2n - 1. \\ xy = n^2 - n - 2. \end{cases}$

8. $\begin{cases} x^2 + xy + y^2 = 63. \\ x - y = 3. \end{cases}$

9. $\begin{cases} x^2 - xy + y^2 = a^2 + 3b^2. \\ x + y = 2a. \end{cases}$

10. $\begin{cases} x^3 + y^3 = 280. \\ x^2 - xy + y^2 = 28. \end{cases}$

11. $\begin{cases} x^2 + xy + y^2 = 7. \\ x^2 - xy + y^2 = 19. \end{cases}$

12. $\begin{cases} d^3 + n^3 = 407. \\ d + n = 11. \end{cases}$

13. $\begin{cases} x^2 + 9y^2 = 50. \\ x - 3y = 0. \end{cases}$

14. $\begin{cases} xy = -16. \\ 2x + y = 14. \end{cases}$

15. $\begin{cases} 36x^2 + 64y^2 = 85. \\ 6x + 8y = 11. \end{cases}$

16. $\begin{cases} x^3 - 8y^3 = 189. \\ x - 2y = 9. \end{cases}$

311. CASE IV. *When each equation is of the second degree, and homogeneous; that is, when each term involving the unknown numbers is of the second degree with respect to them* (§ 65).

Certain equations of this form may be solved by the method of Case I or Case III. (See Exs. 1, § 308, and 2, § 310.)

The method of Case IV should be used only when the example cannot be solved by Cases I or III.

Ex. Solve the equations $\begin{cases} x^2 - 2xy = 5. & (1) \\ x^2 + y^2 = 29. & (2) \end{cases}$

Putting in the given equations $\qquad y = vx,$

we have $\qquad x^2 - 2vx^2 = 5;$ or, $x^2 = \dfrac{5}{1-2v};$ $\qquad (3)$

and $\qquad x^2 + v^2x^2 = 29;$ or, $x^2 = \dfrac{29}{1+v^2}.$

Equating values of x^2, $\qquad \dfrac{5}{1-2v} = \dfrac{29}{1+v^2},$ or $5v^2 + 58v = 24.$

Solving this equation, $\qquad v = \dfrac{2}{5}$ or $-12.$

Substituting these values in (3), we have

$$x^2 = \frac{5}{1-\frac{4}{5}} \text{ or } \frac{5}{1+24} = 25 \text{ or } \frac{1}{5}; \text{ then, } x = \pm 5 \text{ or } \pm\frac{1}{\sqrt{5}}.$$

Substituting the values of v and x in the equation $y = vx$,

$$y = \frac{2}{5}(\pm 5) \text{ or } -12\left(\pm\frac{1}{\sqrt{5}}\right) = \pm 2 \text{ or } \mp\frac{12}{\sqrt{5}}.$$

The solution is $x = \pm 5,\ y = \pm 2;$ or, $x = \pm\dfrac{1}{\sqrt{5}},\ y = \mp\dfrac{12}{\sqrt{5}}.$

In finding y from the equation $y = vx$, care must be taken to multiply each value of x by the value of v which was used to obtain it.

The given equations may also be solved as follows:

Dividing (1) by (2), $\dfrac{x^2 - 2xy}{x^2 + y^2} = \dfrac{5}{29},$ or $29x^2 - 58xy = 5x^2 + 5y^2.$

Then, $\qquad 5y^2 + 58xy - 24x^2 = 0,$ or $(5y - 2x)(y + 12x) = 0.$

Placing $5y - 2x = 0,$ $\qquad y = \dfrac{2x}{5};$ substituting in (1),

$$x^2 - \frac{4x^2}{5} = 5, \text{ or } x^2 = 25.$$

Then, $\qquad x = \pm 5,$ and $y = \dfrac{2x}{5} = \pm 2.$

Placing $y+12x=0$, $\quad y=-12x$; substituting in (1),

$$x^2+24x^2=5, \text{ or } x^2=\frac{1}{5}.$$

Then, $\quad x=\pm\frac{1}{\sqrt{5}}$, and $y=-12x=\mp\frac{12}{\sqrt{5}}$.

EXERCISE 144

Solve the following equations:

1. $\begin{cases} x^2+y^2=25. \\ x^2-xy=4. \end{cases}$

2. $\begin{cases} x^2+3xy=-5. \\ 2xy-y^2=-24. \end{cases}$

3. $\begin{cases} 5x^2-y^2=9. \\ xy-3y^2=-90. \end{cases}$

4. $\begin{cases} x^2+xy+y^2=19. \\ 2x^2+xy=-2. \end{cases}$

5. $\begin{cases} 4x^2-xy-y^2=-16. \\ 3xy+y^2=28. \end{cases}$

6. $\begin{cases} p^2+pq-5q^2=25. \\ p^2+4q^2=40. \end{cases}$

7. $\begin{cases} x^2-2xy-4y^2=-41. \\ x^2-5xy+8y^2=58. \end{cases}$

8. $\begin{cases} 2x^2+7xy+4y^2=2. \\ 3x^2+8xy-4y^2=-72. \end{cases}$

9. $\begin{cases} 4x^2-2xy-y^2=-16. \\ 5x^2-7xy=-36. \end{cases}$

10. $\begin{cases} 3x^2-xy-40y^2=30. \\ 5x^2-3xy-72y^2=38. \end{cases}$

312. Special Methods for the Solution of Simultaneous Equations of Higher Degree.

No general rules can be given for examples which do not come under the cases just considered; various artifices are employed, familiarity with which can only be gained by experience.

1. Solve the equations $\begin{cases} x^3-y^3=19. & (1) \\ x^2y-xy^2=6. & (2) \end{cases}$

Multiply (2) by 3, $\quad 3x^2y-3xy^2=18.$ (3)

Subtract (3) from (1), $x^3-3x^2y+3xy^2-y^3=1.$

Extracting cube roots, $\quad x-y=1.$ (4)

Dividing (2) by (4), $\quad xy=6.$ (5)

Solving equations (4) and (5) by the method of § 310, we find $x=3$, $y=2$; or, $x=-2$, $y=-3$.

2. Solve the equations $\begin{cases} x^3+y^3=9\,xy. \\ x+y=6. \end{cases}$

Putting $x = u + v$ and $y = u - v$,

$(u+v)^3 + (u-v)^3 = 9(u+v)(u-v)$, or $2\,u^3 + 6\,uv^2 = 9(u^2 - v^2)$; (1)

and $\quad (u+v)+(u-v)=6,\ 2\,u=6,$ or $u=3.$

Putting $u = 3$ in (1), $\quad 54 + 18\,v^2 = 9(9 - v^2).$

Whence, $\quad v^2 = 1,$ or $v = \pm 1.$

Therefore, $\quad x = u + v = 3 \pm 1 = 4$ or 2;

and $\quad y = u - v = 3 \mp 1 = 2$ or 4.

The solution is $x = 4,\ y = 2$; or, $x = 2,\ y = 4$.

The artifice of substituting $u + v$ and $u - v$ for x and y is advantageous in any case where the given equations are *symmetrical* (§ 310) with respect to x and y. See also Ex. 4.

3. Solve the equations $\begin{cases} x^2+y^2+2\,x+2\,y=23. & (1) \\ xy=6. & (2) \end{cases}$

Multiplying (2) by 2, $\quad 2\,xy = 12.$ (3)

Add (1) and (3), $\quad x^2 + 2\,xy + y^2 + 2\,x + 2\,y = 35.$

Or, $\quad (x+y)^2 + 2(x+y) = 35.$

Completing the square, $(x+y)^2 + 2(x+y) + 1 = 36.$

Then, $\quad (x+y)+1 = \pm 6$; and $x+y=5$ or -7. (4)

$$\begin{array}{ll} \text{Squaring (4),} & x^2+2\,xy+y^2=25 \text{ or } 49. \\ \text{Multiplying (2) by 4,} & \underline{4\,xy \qquad\quad = 24.\qquad\quad} \\ \text{Subtracting,} & x^2-2\,xy+y^2=1 \text{ or } 25. \end{array}$$

Whence, $\quad x - y = \pm 1$ or ± 5. (5)

Adding (4) and (5), $\quad 2\,x = 5 \pm 1$, or -7 ± 5.

Whence, $\quad x = 3,\ 2,\ -1$, or -6.

Subtracting (5) from (4), $\quad 2\,y = 5 \mp 1$, or -7 ∓ 5.

Whence, $\quad y = 2,\ 3,\ -6$, or -1.

The solution is $x=3,\ y=2$; $x=2,\ y=3$; $x=-1,\ y=-6$; or $x=-6,\ y=-1$.

4. Solve the equations $\begin{cases} x^4+y^4=97. \\ x+y=-1. \end{cases}$

Putting $x=u+v$ and $y=u-v$,

$$(u+v)^4+(u-v)^4=97, \text{ or } 2u^4+12u^2v^2+2v^4=97, \quad (1)$$

and $$(u+v)+(u-v)=-1,\ 2u=-1, \text{ or } u=-\frac{1}{2}.$$

Substituting value of u in (1), $\frac{1}{8}+3v^2+2v^4=97.$

Solving this, $$v^2=\frac{25}{4} \text{ or } -\frac{31}{4}; \text{ and } v=\pm\frac{5}{2} \text{ or } \pm\frac{\sqrt{-31}}{2}.$$

Then, $x=u+v=-\frac{1}{2}\pm\frac{5}{2}$, or $-\frac{1}{2}\pm\frac{\sqrt{-31}}{2}=2,\ -3$, or $\frac{-1\pm\sqrt{-31}}{2}$;

and, $y=u-v=-\frac{1}{2}\mp\frac{5}{2}$, or $-\frac{1}{2}\mp\frac{\sqrt{-31}}{2}=-3,\ 2$, or $\frac{-1\mp\sqrt{-31}}{2}$.

The solution is $x=2,\ y=-3$; $x=-3,\ y=2$; $x=\frac{-1+\sqrt{-31}}{2}$, $y=\frac{-1-\sqrt{-31}}{2}$; or $x=\frac{-1-\sqrt{-31}}{2}$, $y=\frac{-1+\sqrt{-31}}{2}$.

MISCELLANEOUS AND REVIEW EXAMPLES

EXERCISE 145

Solve the following equations:

1. $\begin{cases} 2xy+x=-36. \\ xy-3y=-5. \end{cases}$

2. $\begin{cases} x^2+y^2+x-y=32. \\ xy=6. \end{cases}$

3. $\begin{cases} g^2+h^2=\frac{289}{36}. \\ gh=\frac{10}{3}. \end{cases}$

4. $\begin{cases} x^2-3xy-4y^2=0. \\ 3x-5y=46. \end{cases}$

5. $\begin{cases} x^2-2y^2+3x=-8. \\ x^2-2y^2-4y=-2. \end{cases}$

6. $\begin{cases} \frac{1}{x^2}+\frac{1}{y^2}=74. \\ \frac{1}{x}-\frac{1}{y}=12. \end{cases}$

7. $\begin{cases} \frac{2x}{y}-5x=\frac{11}{2}. \\ \frac{3y}{x}+4y=\frac{2}{3}. \end{cases}$

8. $\begin{cases} \frac{x}{y}+\frac{y}{x}=-\frac{10}{3}. \\ x-y=1. \end{cases}$

9. $\begin{cases} x-\frac{2}{y}=-\frac{a}{b}. \\ y+\frac{2}{x}=\frac{3b}{a}. \end{cases}$

10. $\begin{cases} x^4+y^4=17. \\ x-y=3. \end{cases}$

11. $\begin{cases} 4d+k-3dk=-6. \\ d-5k+2dk=10. \end{cases}$

12. $\begin{cases} \dfrac{1}{x^2}+\dfrac{1}{xy}+\dfrac{1}{y^2}=49. \\ \dfrac{1}{x}+\dfrac{1}{y}=8. \end{cases}$

13. $\begin{cases} x+y=35. \\ \sqrt[3]{x}+\sqrt[3]{y}=5. \end{cases}$

14. $\begin{cases} 11x^2-xy-y^2=45. \\ 7x^2+3xy-2y^2=20. \end{cases}$

15. $\begin{cases} x^2+4xy=13. \\ 2xy+9y^2=87. \end{cases}$

16. $\begin{cases} x^2y^2-24xy+95=0. \\ 3x-2y=-13. \end{cases}$

17. $\begin{cases} 3x^2-5xy=2a^2+13ab-7b^2. \\ x+y=3(a-b). \end{cases}$

18. $\begin{cases} \dfrac{3x+2y}{3x-2y}+\dfrac{3x-2y}{3x+2y}=\dfrac{41}{20}. \\ 8y^2+3x^2=29. \end{cases}$

19. $\begin{cases} \dfrac{1}{xy}=6a^2. \\ x+y=5axy. \end{cases}$

20. $\begin{cases} \dfrac{1}{x^3}+\dfrac{1}{y^3}=-19a^3. \\ \dfrac{1}{x}+\dfrac{1}{y}=-a. \end{cases}$

21. $\begin{cases} e^2+9t^2+4e=9. \\ et+2t=-2. \end{cases}$

22. $\begin{cases} x^3+y^3=2a^3+24a. \\ x^2y+xy^2=2a^3-8a. \end{cases}$

23. $\begin{cases} \sqrt{2x^2-9}=3y+6. \\ \sqrt{x^4-17y^2}=x^2-5. \end{cases}$

24. $\begin{cases} 3x^2-xy-xz=4. \\ 5x-2y=1. \\ 4x+3z=-5. \end{cases}$

25. $\begin{cases} x^2y+xy^2=56. \\ x+y=-1. \end{cases}$

26. $\begin{cases} \dfrac{x^2}{y}+\dfrac{y^2}{x}=\dfrac{19}{6}. \\ \dfrac{1}{x}+\dfrac{1}{y}=\dfrac{1}{6}. \end{cases}$

27. $\begin{cases} 3x^2+3y^2=10xy. \\ \dfrac{1}{x}+\dfrac{1}{y}=\dfrac{4}{3}. \end{cases}$

28. $\begin{cases} x^2y+y^2x=42. \\ \dfrac{1}{x}+\dfrac{1}{y}=\dfrac{7}{6}. \end{cases}$

29. $\begin{cases} 5q^2+qs-3s^2=27. \\ 4q^2-4qs+3s^2=72. \end{cases}$

30. $\begin{cases} y^2+4xy-3y=42. \\ 2y^2-xy+5y=-10. \end{cases}$

31. $\begin{cases} 16x^2y^2-104xy=-105. \\ x-y=-2. \end{cases}$

32.* $\begin{cases} x^4+x^2y^2+y^4=481. \\ x^2-xy+y^2=37. \end{cases}$

33. $\begin{cases} 9x^2-13xy-3x=-123. \\ xy+4y^2+2y=125. \end{cases}$

* Divide the first equation by the second.

34. $\begin{cases} x^4+y^4=257. \\ x+y=3. \end{cases}$

35. $\begin{cases} \dfrac{2}{x^2}+\dfrac{1}{xy}+\dfrac{5}{y^2}=44. \\ \dfrac{1}{xy}-\dfrac{1}{y^2}=-12. \end{cases}$

36. $\begin{cases} 2x^2+y^2-z^2=43. \\ x-3y+z=17. \\ x+y-3z=13. \end{cases}$

37. $\begin{cases} x^2-xy+y^2=\frac{7}{324}. \\ x-y=\frac{1}{18}. \end{cases}$

38. $\begin{cases} (x+y)+xy=11. \\ (x+y)^2+x^2y^2=61. \end{cases}$

39. $\begin{cases} xy-(x-y)=-9. \\ xy(x-y)=-20. \end{cases}$

40. $\begin{cases} x^2+y^2=xy+1. \\ x^4+y^4=x^2y^2+1. \end{cases}$

41. $\begin{cases} x^5+y^5=211. \\ x+y=1. \end{cases}$

42. $\begin{cases} x^2y-x=-6. \\ x^6y^3-x^3=-72. \end{cases}$

43. $\begin{cases} \dfrac{1}{x^2}+\dfrac{1}{xy}+\dfrac{1}{y^2}=\dfrac{a^4+a^2b^2+b^4}{a^2b^2}. \\ \dfrac{1}{x^2}-\dfrac{1}{xy}+\dfrac{1}{y^2}=\dfrac{a^4-a^2b^2+b^4}{a^2b^2}. \end{cases}$

PROBLEMS IN PHYSICS

1. From the equations $v=gt$ and $S=\frac{1}{2}gt^2$, find v in terms of S and g.

2. From the equations $C=\dfrac{E}{R}$ and $EC=\dfrac{H}{t}$, find H in terms of C, R, and t.

3. From the equations $E=FS$, $F=ma$, $S=\frac{1}{2}at^2$, and $v=at$, find E in terms of m and v.

313. Problems involving Simultaneous Equations of Higher Degree.

In solving problems which involve simultaneous equations of higher degree, only those solutions should be retained which satisfy the conditions of the problem. (Compare § 176.)

EXERCISE 146

1. The difference of the squares of two numbers is 56, and the difference of the numbers is $\frac{2}{7}$ their sum. Find the numbers.

2. The sum of the squares of two numbers is 61, and the product of their squares is 900. Find the numbers.

3. The product of the sum of two numbers by the smaller is 21, and the product of their difference by the greater is 4. Find the numbers.

4. The sum of the cubes of two numbers is 224; and if the product of the numbers be subtracted from the sum of their squares, the remainder is 28. Find the numbers.

5. Two numbers are expressed by the same two digits in reverse order. The sum of the numbers equals the square of the sum of the digits, and the difference of the numbers equals 5 times the square of the smaller digit. Find the numbers.

6. The square of the sum of two numbers exceeds their product by 84; and the sum of the numbers, plus the square root of their product, equals 14. Find the numbers.

7. The difference of the cubes of two numbers is 342; and if the product of the numbers be multiplied by their difference, the result is 42. Find the numbers.

8. A party at a hotel spent a certain sum. Had there been 5 more, and each had spent 50 cents less, the bill would have been $24.75. Had there been 3 fewer, and each had spent 50 cents more, the bill would have been $9.75. How many were there, and what did each spend?

9. The simple interest of $700, for a certain number of years, at a certain rate, is $182. If the time were 4 years less, and the rate $1\frac{1}{2}\%$ more, the interest would be $133. Find the time and the rate.

10. If the digits of a number of two figures be reversed, the quotient of this number by the given number is $1\frac{3}{4}$, and their product 1008. Find the number.

11. The square of the smaller of two numbers, added to twice their product, gives 7 times the smaller number; and the square of the greater exceeds the product of the numbers by 6 times the smaller number. Find the numbers.

12. A rectangular piece of cloth, when wet, shrinks one-sixth in its length, and one-twelfth in its width. If the area is diminished by $12\frac{3}{4}$ square feet, and the length of the four sides by $6\frac{1}{2}$ feet, find the original dimensions.

13. A and B travel from P to Q, 14 miles, at uniform rates, B taking one-third of an hour longer than A to perform the journey. On the return, each travels one mile an hour faster, and B now takes one-fourth of an hour longer than A. Find their rates of travelling.

14. A and B run a race of two miles, B winning by two minutes. A now increases his speed by two miles an hour, and B diminishes his by the same amount, and A wins by two minutes. Find their original rates.

15. A man ascends the last half of a mountain at a rate one-half mile an hour less than his rate during the first half, and reaches the top in $3\frac{3}{8}$ hours. On the descent, his rate is one mile an hour greater than during the first half of the ascent, and he accomplishes it in $2\frac{1}{2}$ hours. Find the distance to the top, and his rate during the first half of the ascent.

16. The square of the second digit of a number of three digits exceeds twice the sum of the first and third by 3. The sum of the first and second digits exceeds 4 times the third by 1; and if 495 be subtracted from the number, the digits will be reversed. Find the number.

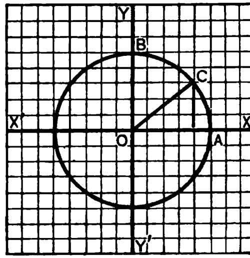
17. A ship has provisions for 36 days. If the crew were 16 greater, and the daily ration one-half pound less, the provisions would last 30 days; if the crew were 2 fewer, and the daily ration one pound greater, they would last 24 days. Find the number of men, and the daily ration.

18. A man lends $2100 in two amounts, at different rates of interest, and the two sums produce equal returns. If the first portion had been loaned at the second rate, it would have produced $48; and if the second portion had been loaned at the first rate, it would have produced $27. Find the rates.

19. A can do a piece of work in 2 hours less time than B; and together they can do the work in $1\frac{1}{8}$ hours less time than A alone. How long does each alone take to do the work?

GRAPHICAL REPRESENTATION OF SIMULTANEOUS QUADRATIC EQUATIONS WITH TWO UNKNOWN NUMBERS

314. 1. Consider the equation $x^2 + y^2 = 25$.



This means that, for any point on the graph, the square of the abscissa, plus the square of the ordinate, equals 25.

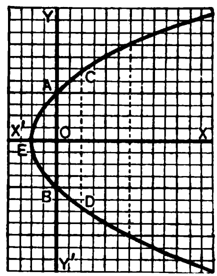
But the square of the abscissa of any point, plus the square of the ordinate, equals the square of the distance of the point from the origin; for the distance is the hypotenuse of a right triangle, whose other two sides are the abscissa and ordinate.

Then the square of the distance from O of any point on the graph is 25; or, the distance from O of any point on the graph is 5.

Thus, the graph is a circle of radius 5, having its centre at O.

(The graph of any equation of the form $x^2 + y^2 = a$ is a circle.)

2. Consider the equation $y^2 = 4x + 4$.



Scale: ¼ inch.

If $x = 0$, $\quad y^2 = 4$, or $y = \pm 2$. $\qquad (A, B)$

If $x = 1$, $\quad y^2 = 8$, or $y = \pm 2\sqrt{2}$. $\qquad (C, D)$

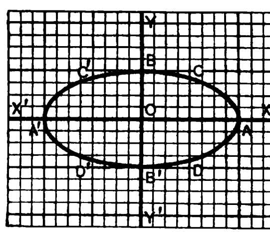
If $x = -1$, $y = 0$. Etc. $\qquad (E)$

The graph extends indefinitely to the right of YY'.

If x is negative and < -1, y^2 is negative, and therefore y *imaginary;* then, no part of the graph lies to the left of E.

(The graph of Ex. 2 is a *parabola;* as also is the graph of any equation of the form $y^2 = ax$ or $y^2 = ax + b$. The graphs of §§ 303 and 305 are parabolas.)

3. Consider the equation $x^2 + 4y^2 = 4$.



Scale: ¼ inch.

In this case it is convenient to first locate the points where the graph intersects the axes.

If $y = 0$, $\quad x^2 = 4$, or $x = \pm 2$. $\qquad (A, A')$

If $x = 0$, $4y^2 = 4$, or $y = \pm 1$. $\qquad (B, B')$

Putting $x = \pm 1$, $4y^2 = 3$, $y^2 = \frac{3}{4}$, or

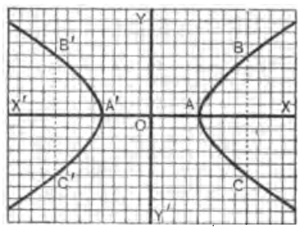
$y = \pm \frac{\sqrt{3}}{2}$. $\qquad (C, D, C', D')$

If x has any value >2, or <-2, y^2 is negative, and y imaginary; then, no part of the graph lies to the right of A, or left of A'.

If y has any value >1, or <-1, x^2 is negative, and x imaginary; then, no part of the graph lies above B, or below B'.

(The graph of Ex. 3 is an *ellipse*; as also is the graph of any equation of the form $ax^2 + by^2 = c$.)

4. Consider the equation $x^2 - 2y^2 = 1$.



Scale: ¼ inch.

Here $x^2 - 1 = 2y^2$, or $y^2 = \dfrac{x^2 - 1}{2}$.

If $x = \pm 1$, $y^2 = 0$, or $y = 0$. (A', A)

If x has any value between 1 and -1, y^2 is negative, and y imaginary.

Then, no part of the graph lies between A and A'.

If $x = \pm 2$, $y^2 = \dfrac{3}{2}$; or $y = \pm\sqrt{\dfrac{3}{2}}$. ($B$, C, B', C')

The graph has two branches, BAC and $B'A'C'$, each of which extends to an indefinitely great distance from O.

(The graph of Ex. 4 is a *hyperbola*; as also is the graph of any equation of the form $ax^2 - by^2 = c$, or $xy = a$.)

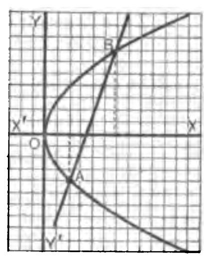
EXERCISE 147

Plot the graphs of the following:

1. $xy = -6$.
2. $x^2 = 3y$.
3. $x^2 + y^2 = 4$.
4. $y^2 = 5x - 1$.
5. $4x^2 + 9y^2 = 36$.
6. $4x^2 - y^2 = -4$.

315. Graphical Representation of Solutions of Simultaneous Quadratic Equations.

1. Consider the equations $\begin{cases} y^2 = 4x. \\ 3x - y = 5. \end{cases}$



Scale: ¼ inch.

The graph of $y^2 = 4x$ is the parabola AOB.

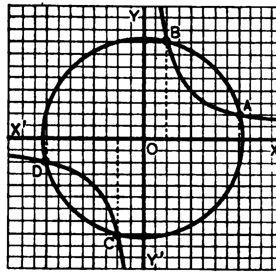
The graph of $3x - y = 5$ is the straight line AB, intersecting the parabola at the points A and B, respectively.

To find the co-ordinates of A and B, we proceed as in § 184; that is, we *solve the given equations*.

The solution is $x = 1$, $y = -2$; or, $x = \dfrac{25}{9}$, $y = \dfrac{10}{3}$ (§ 309).

It may be verified in the figure that these are the co-ordinates of A and B, respectively.

Hence, *if any two graphs intersect, the co-ordinates of any point of intersection form a solution of the set of equations represented by the graphs.*



Scale: $\frac{1}{8}$ inch.

2. Consider the equations

$$\begin{cases} x^2+y^2=17. \\ xy=4. \end{cases}$$

The graph of $x^2+y^2=17$ is the circle AD, whose centre is at O, and radius $\sqrt{17}$.

The graph of $xy=4$ is a hyperbola, having its branches in the angles XOY and $X'OY'$, respectively, and intersecting the circle at the points A and B in angle XOY, and at the points C and D in angle $X'OY'$.

The solution of the given equations is (§ 310),

$x=4,\ y=1$; $x=1,\ y=4$; $x=-1,\ y=-4$; and $x=-4,\ y=-1$.

It may be verified in the figure that these are the co-ordinates of A, B, C, and D, respectively.

EXERCISE 148

Find the graphs of the following sets of equations, and in each case verify the principle of § 315:

1. $\begin{cases} x^2+4y^2=4. \\ x-y=1. \end{cases}$

2. $\begin{cases} x^2-4y=-7. \\ 2y+3y=4. \end{cases}$

3. $\begin{cases} 9x^2+y^2=148. \\ xy=-8. \end{cases}$

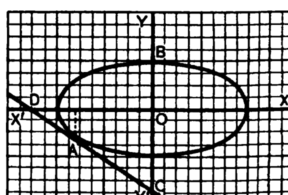
4. $\begin{cases} x^2+y^2=29. \\ xy=10. \end{cases}$

5. $\begin{cases} 2x^2+5y^2=53. \\ 3x^2-4y^2=-24. \end{cases}$

6. $\begin{cases} x^2+y^2=13. \\ 4x-9y=6. \end{cases}$

316. 1. Consider the equations

$$\begin{cases} x^2+4y^2=4. & (1) \\ 2x+3y=-5. & (2) \end{cases}$$



Scale: $\frac{1}{4}$ inch.

The graph of $x^2+4y^2=4$ is the ellipse AB.

The graph of $2x+3y=-5$ is the straight line CD.

To solve the given equations, we have, from (2), $x = \frac{-3y-5}{2}$.

Substituting in (1), $\frac{9y^2+30y+25}{4} + 4y^2 = 4$.

Then, $25y^2+30y+9=0$, or $(5y+3)(5y+3)=0$.

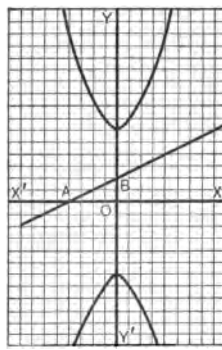
This equation has *equal roots;* the only value of y is $-\frac{3}{5}$; and $x = -\frac{8}{5}$.

The line has but one point in common with the ellipse, and is tangent to it.

Then, *if the equation obtained by eliminating one of the unknown numbers has equal roots, the graphs are tangent to each other.*

2. Consider the equations

$$\begin{cases} 9x^2 - y^2 = -9. & (1) \\ x - 2y = -2. \end{cases}$$



Scale: $\frac{1}{8}$ inch.

The graph of $9x^2 - y^2 = -9$ is a hyperbola, having its branches above and below XX' respectively.

The graph of $x - 2y = -2$ is the straight line AB.

To solve the given equations, we substitute $x = 2y - 2$ in (1).

Then, $9(4y^2 - 8y + 4) - y^2 = -9$,

or $35y^2 - 72y + 45 = 0$.

This equation has *imaginary roots*, which shows that the line does not intersect the hyperbola.

In general, *if the equation obtained by eliminating one of the unknown numbers has imaginary roots, the graphs do not intersect.*

EXERCISE 149

Find the graphs of the following sets of equations, and in each case verify the principles of § 316:

1. $\begin{cases} x^2 + y^2 = 4. \\ 3x - y = 8. \end{cases}$

2. $\begin{cases} y^2 - 3x = -3. \\ x + 2y = -2. \end{cases}$

3. $\begin{cases} x^2 - y^2 = 9. \\ 5x - 4y = -9. \end{cases}$

4. $\begin{cases} x^2 + y^2 = 1. \\ 2y^2 - 3x = 5. \end{cases}$

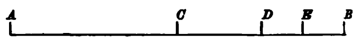
XXIII. VARIABLES AND LIMITS

317. A *variable number*, or simply a *variable*, is a number which may assume, under the conditions imposed upon it, an indefinitely great number of different values.

A *constant* is a number which remains unchanged throughout the same discussion.

318. A *limit* of a variable is a constant number, the difference between which and the variable may be made less than any assigned number, however small.

Suppose, for example, that a point moves from A towards B under the condition that it shall move, during successive equal intervals of time, first from A to C, half-way between A and B; then to D, half-way between C and B; then to E, half-way between D and B; and so on indefinitely.



In this case, the distance between the moving point and B can be made less than any assigned number, however small.

Hence, the distance from A to the moving point is a variable which approaches the constant value AB as a limit.

Again, the distance from the moving point to B is a variable which approaches the limit 0.

319. Interpretation of $\frac{a}{0}$.

Consider the series of fractions $\frac{a}{3}, \frac{a}{.3}, \frac{a}{.03}, \frac{a}{.003}, \cdots$.

Here each denominator after the first is one-tenth of the preceding denominator.

It is evident that, by sufficiently continuing the series, the denominator may be made less than any assigned number, however small, and the value of the fraction greater than any assigned number, however great.

In other words,

If the numerator of a fraction remains constant, while the denominator approaches the limit 0, *the value of the fraction increases without limit.*

It is customary to express this principle as follows:

$$\frac{a}{0}=\infty.$$

The symbol ∞ is called *Infinity;* it simply stands for that which is greater than any number, however great, and has no fixed value.

320. Interpretation of $\frac{a}{\infty}$.

Consider the series of fractions $\frac{a}{3}, \frac{a}{30}, \frac{a}{300}, \frac{a}{3000}, \cdots$.

Here each denominator after the first is ten times the preceding denominator.

It is evident that, by sufficiently continuing the series, the denominator may be made greater than any assigned number, however great, and the value of the fraction less than any assigned number, however small.

In other words,

If the numerator of a fraction remains constant, while the denominator increases without limit, the value of the fraction approaches the limit 0.

It is customary to express this principle as follows:

$$\frac{a}{\infty}=0.$$

321. No *literal meaning* can be attached to such results as

$$\frac{a}{0}=\infty, \text{ or } \frac{a}{\infty}=0;$$

for there can be no such thing as division unless the divisor is a *finite number.*

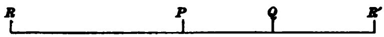
If such forms occur in mathematical investigations, they must be interpreted as indicated in §§ 319 and 320. (Compare § 420.)

THE PROBLEM OF THE COURIERS

322. The following discussion will further illustrate the form $\frac{a}{0}$, besides furnishing an interpretation of the form $\frac{0}{0}$.

The Problem of the Couriers.

Two couriers, A and B, are travelling along the same road in the same direction, RR', at the rates of m and n miles an hour, respectively. If at any time, say 12 o'clock, A is at P, and B is a miles beyond him at Q, after how many hours, and how many miles beyond P, are they together?



Let A and B meet x hours after 12 o'clock, and y miles beyond P.

They will then meet $y - a$ miles beyond Q.

Since A travels mx miles, and B nx miles, in x hours, we have

$$\begin{cases} y = mx, \\ y - a = nx. \end{cases}$$

Solving these equations, we obtain

$$x = \frac{a}{m-n}, \text{ and } y = \frac{am}{m-n}.$$

We will now discuss these results under different hypotheses.

1. $m > n$.

In this case, the values of x and y are *positive*.

This means that the couriers meet at some time *after* 12, at some point to the *right* of P.

This agrees with the hypothesis made; for if m is greater than n, A is travelling faster than B; and he must overtake him at some point beyond their positions at 12 o'clock.

2. $m < n$.

In this case, the values of x and y are *negative*.

This means that the couriers met at some time *before* 12, at some point to the *left* of P. (Compare § 16.)

This agrees with the hypothesis made; for if m is less than n, A is travelling more slowly than B; and they must have been together before 12 o'clock, and before they could have advanced as far as P.

3. $a = 0$, and $m > n$ or $m < n$.

In this case, $x = 0$ and $y = 0$.

This means that the travellers are together at 12 o'clock, at the point P.

This agrees with the hypothesis made; for if $a = 0$, and m and n are unequal, the couriers are together at 12 o'clock, and are travelling at unequal rates; and they could not have been together before 12, and will not be together afterwards.

4. $m = n$, and a not equal to 0.

In this case, the values of x and y take the forms $\frac{a}{0}$ and $\frac{am}{0}$, respectively.

If $m - n$ approaches the limit 0, the values of x and y increase without limit (§ 319); hence, if $m = n$, no fixed values can be assigned to x and y, and the problem is impossible.

In this case, *the result in the form* $\frac{a}{0}$ *indicates that the given problem is impossible.*

This agrees with the hypothesis made; for if $m = n$, and a is not zero, the couriers are a miles apart at 12 o'clock, and are travelling at the same rate; and they never could have been, and never will be together.

5. $m = n$, and $a = 0$.

In this case, the values of x and y take the form $\frac{0}{0}$.

If $a = 0$, and $m = n$, the couriers are together at 12 o'clock, and travelling at the same rate.

Hence, they always have been, and always will be, together.

In this case, the number of solutions is indefinitely great; for any value of x whatever, together with the corresponding value of y, will satisfy the given conditions.

In this case, *the result in the form* $\frac{0}{0}$ *indicates that the number of solutions is indefinitely great.*

XXIV. INDETERMINATE EQUATIONS

323. It was shown, in § 163, that a single equation involving two or more unknown numbers is satisfied by an indefinitely great number of sets of values of these numbers.

If, however, the unknown numbers are required to satisfy other conditions, the number of solutions may be finite.

We shall consider in the present chapter the solution of indeterminate linear equations, in which the unknown numbers are restricted to *positive integral* values.

324. Solution of Indeterminate Linear Equations in Positive Integers.

1. Solve $7x + 5y = 118$ in positive integers.

Dividing by 5, the smaller of the two coefficients,

$$x + \frac{2x}{5} + y = 23 + \frac{3}{5}; \text{ or, } \frac{2x-3}{5} = 23 - x - y.$$

Since, by the conditions of the problem, x and y must be positive integers, $\frac{2x-3}{5}$ must be an integer.

Let this integer be represented by p.

Then, $$\frac{2x-3}{5} = p, \text{ or } 2x - 3 = 5p. \quad (1)$$

Dividing (1) by 2, $x - 1 - \frac{1}{2} = 2p + \frac{p}{2}$; or, $x - 1 - 2p = \frac{p+1}{2}$.

Since x and p are integers, $x - 1 - 2p$ is an integer; and therefore $\frac{p+1}{2}$ must be an integer.

Let this integer be represented by q.

Then, $$\frac{p+1}{2} = q, \text{ or } p = 2q - 1.$$

Substituting in (1), $$2x - 3 = 10q - 5.$$

Whence, $$x = 5q - 1. \quad (2)$$

Substituting this value in the given equation,

$$35q - 7 + 5y = 118;\ \text{or,}\ y = 25 - 7q. \qquad (3)$$

Equations (2) and (3) form the *general solution in integers* of the given equation.

By giving to q the value zero, or any positive or negative integer, we shall obtain sets of integral values of x and y which satisfy the given equation.

If q is zero, or any negative integer, x will be negative.

If q is any positive integer >3, y will be negative.

Hence, the only *positive integral* values of x and y which satisfy the given equation are those obtained from the values 1, 2, 3 of q.

That is, $x = 4,\ y = 18;\ x = 9,\ y = 11;$ and $x = 14,\ y = 4$.

2. Solve $5x - 7y = 11$ in *least* positive integers.

Dividing by 5, the coefficient of smaller absolute value,

$$x - y - \frac{2y}{5} = 2 + \frac{1}{5};\ \text{or,}\ x - y - 2 = \frac{2y+1}{5}.$$

Then, $\frac{2y+1}{5}$ must be an integer.

Let $\frac{2y+1}{5} = p$; or, $2y + 1 = 5p$.

Dividing by 2, $y + \frac{1}{2} = 2p + \frac{p}{2}$; or, $y - 2p = \frac{p-1}{2}$.

Then, $\frac{p-1}{2}$ must be an integer.

Let $\frac{p-1}{2} = q$; or, $p = 2q + 1$.

Then, $y = \frac{5p-1}{2} = \frac{10q+5-1}{2} = 5q + 2$.

Then, from the given equation, $x = \frac{7y+11}{5} = 7q + 5$.

The solution in least positive integers is when $q=0$; that is, $x=5$, $y=2$.

3. In how many ways can the sum of $15 be paid with dollars, half-dollars, and dimes, the number of dimes being equal to the number of dollars and half-dollars together?

Let $x =$ number of dollars,

$y =$ number of half-dollars,

and $z =$ number of dimes.

By the conditions, $\begin{cases} 10x + 5y + z = 150, \\ x + y = z. \end{cases}$ (1)

Adding, $11x + 6y + z = 150 + z,$

or, $11x + 6y = 150.$ (2)

Dividing by 6, $x + \frac{5x}{6} + y = 25.$

Then, $\frac{5x}{6}$ must be an integer; or, x must be a multiple of 6.

Let $x = 6p$, where p is an integer.

Substitute in (2), $66p + 6y = 150$, or $y = 25 - 11p.$

Substitute in (1), $z = 6p + 25 - 11p = 25 - 5p.$

The only positive integral solutions are when $p = 1$ or 2.

Then, the number of ways is two; either 6 dollars, 14 half-dollars, and 20 dimes; or, 12 dollars, 3 half-dollars, and 15 dimes.

EXERCISE 150

Solve the following in positive integers:

1. $3x + 5y = 29.$
2. $7x + 2y = 39.$
3. $6x + 29y = 274.$
4. $4x + 31y = 473.$
5. $42x + 11y = 664.$
6. $10x + 7y = 297.$
7. $23x + 9y = 151.$
8. $8x + 71y = 1933.$
9. $\begin{cases} 8x - 11y + 2z = 10. \\ 2x - 9y + z = -8. \end{cases}$
10. $\begin{cases} 3x - 3y + 7z = 101. \\ 4x + 2y - 3z = 5. \end{cases}$

Solve the following in least positive integers:

11. $6x - 7y = 4.$
12. $5x - 8y = 17.$
13. $14x - 5y = 64.$
14. $8x - 31y = 10.$
15. $30x - 13y = 115.$
16. $15x - 38y = -47.$

17. In how many different ways can $1.65 be paid with quarter-dollars and dimes?

18. In how many different ways can 41 shillings be paid with half-crowns, worth $2\frac{1}{2}$ shillings each, and two-shilling pieces?

19. Find two fractions whose denominators are 5 and 7, respectively, whose numerators are the smallest possible positive integers, and whose difference is $\frac{17}{35}$.

20. In how many different ways can $7.15 be paid with fifty-cent, twenty-five cent, and twenty-cent pieces, so that twice the number of fifty-cent pieces, plus twice the number of twenty-cent pieces, shall exceed the number of twenty-five cent pieces by 31?

21. A farmer purchased a certain number of pigs, sheep, and calves for $138. The pigs cost $4 each, the sheep $7 each, and the calves $9 each; and the whole number of animals purchased was 23. How many of each did he buy?

22. In how many ways can $10.00 be paid with twenty-five cent, twenty-cent, and five-cent pieces, so that 3 times the number of twenty-five cent pieces, plus 15 times the number of twenty-cent pieces, shall exceed the number of five-cent pieces by 33?

XXV. RATIO AND PROPORTION

RATIO

325. The **Ratio** of one number a to another number b is the quotient of a divided by b.

Thus, the ratio of a to b is $\frac{a}{b}$; it is also expressed $a:b$.

In the ratio $a:b$, a is called the *first term*, or *antecedent*, and b the *second term*, or *consequent*.

If a and b are positive numbers, and $a > b$, $\frac{a}{b}$ is called a *ratio of greater inequality;* if a is $< b$, it is called a *ratio of less inequality.*

326. *A ratio of greater inequality is decreased, and one of less inequality is increased, by adding the same positive number to each of its terms.*

Let a and b be positive numbers, a being $> b$, and x a positive number.

Since $a > b$, $\qquad ax > bx. \qquad$ (§ 195)

Adding ab to both members (§ 192),

$$ab + ax > ab + bx, \text{ or } a(b + x) > b(a + x).$$

Dividing both members by $b(b + x)$, we have

$$\frac{a}{b} > \frac{a + x}{b + x}. \qquad (\S\ 195)$$

In like manner, if a is $< b$, $\frac{a}{b} < \frac{a + x}{b + x}$.

PROPORTION

327. A **Proportion** is an equation whose members are equal ratios.

Thus, if $a:b$ and $c:d$ are equal ratios,

$$a:b=c:d, \text{ or } \frac{a}{b}=\frac{c}{d},$$

is a proportion.

328. In the proportion $a:b=c:d$, a is called the *first term*, b the *second*, c the *third*, and d the *fourth*.

The first and third terms of a proportion are called the *antecedents*, and the second and fourth terms the *consequents*.

The first and fourth terms are called the *extremes*, and the second and third terms the *means*.

329. If the means of a proportion are equal, either mean is called the **Mean Proportional** between the first and last terms, and the last term is called the **Third Proportional** to the first and second terms.

Thus, in the proportion $a:b=b:c$, b is the mean proportional between a and c, and c is the third proportional to a and b.

The **Fourth Proportional** to three numbers is the fourth term of a proportion whose first three terms are the three numbers taken in their order.

Thus, in the proportion $a:b=c:d$, d is the fourth proportional to a, b, and c.

330. A **Continued Proportion** is a series of equal ratios, in which each consequent is the same as the next antecedent; as,

$$a:b=b:c=c:d=d:e.$$

PROPERTIES OF PROPORTIONS

331. *In any proportion, the product of the extremes is equal to the product of the means.*

Let the proportion be $a:b=c:d$.

Then by § 327, $\frac{a}{b}=\frac{c}{d}.$

Clearing of fractions, $ad=bc.$

332. From the equation $ad = bc$ (§ 331), we obtain

$$a = \frac{bc}{d},\ b = \frac{ad}{c},\ c = \frac{ad}{b},\ \text{and}\ d = \frac{bc}{a}.$$

That is, *in any proportion, either extreme equals the product of the means divided by the other extreme; and either mean equals the product of the extremes divided by the other mean.*

333. (Converse of § 331.) *If the product of two numbers be equal to the product of two others, one pair may be made the extremes, and the other pair the means, of a proportion.*

Let $ad = bc.$

Dividing by bd, $\frac{ad}{bd} = \frac{bc}{bd}$, or $\frac{a}{b} = \frac{c}{d}.$

Whence, by § 327, $a : b = c : d.$

In like manner, we may prove that

$$a : c = b : d,$$

$$c : d = a : b,\ \text{etc.}$$

334. *In any proportion, the terms are in proportion by* **Alternation**; *that is, the means can be interchanged.*

Let the proportion be $a : b = c : d.$

Then by § 331, $ad = bc.$

Whence, by § 333, $a : c = b : d.$

In like manner, it may be proved that the *extremes* can be interchanged.

335. *In any proportion, the terms are in proportion by* **Inversion**; *that is, the second term is to the first as the fourth term is to the third.*

Let the proportion be $a : b = c : d.$

Then, by § 331, $ad = bc.$

Whence, by § 333, $b : a = d : c.$

It follows from § 335 that, in any proportion, the means can be written as the extremes, and the extremes as the means.

336. *The mean proportional between two numbers is equal to the square root of their product.*

Let the proportion be $a : b = b : c$.

Then by § 331, $b^2 = ac$, and $b = \sqrt{ac}$.

337. *In any proportion, the terms are in proportion by* **Composition**; *that is, the sum of the first two terms is to the first term as the sum of the last two terms is to the third term.*

Let the proportion be $a : b = c : d$.

Then, $ad = bc$.

Adding each member of the equation to ac,

$$ac + ad = ac + bc, \text{ or } a(c + d) = c(a + b).$$

Then by § 333, $a + b : a = c + d : c$.

We may also prove $a + b : b = c + d : d$.

338. *In any proportion, the terms are in proportion by* **Division**; *that is, the difference of the first two terms is to the first term as the difference of the last two terms is to the third term.*

Let the proportion be $a : b = c : d$.

Then, $ad = bc$.

Subtracting each member of the equation from ac,

$$ac - ad = ac - bc, \text{ or } a(c - d) = c(a - b).$$

Then, $a - b : a = c - d : c$.

We may also prove $a - b : b = c - d : d$.

339. *In any proportion, the terms are in proportion by* **Composition and Division**; *that is, the sum of the first two terms is to their difference as the sum of the last two terms is to their difference.*

Let the proportion be $a : b = c : d$.

Then by § 337, $$\frac{a+b}{a}=\frac{c+d}{c}. \quad (1)$$

And by § 338, $$\frac{a-b}{a}=\frac{c-d}{c}. \quad (2)$$

Dividing (1) by (2), $$\frac{a+b}{a-b}=\frac{c+d}{c-d}.$$

Whence, $$a+b:a-b=c+d:c-d.$$

340. *In any proportion, if the first two terms be multiplied by any number, as also the last two, the resulting numbers will be in proportion.*

Let the proportion be $\frac{a}{b}=\frac{c}{d}$; then, $\frac{ma}{mb}=\frac{nc}{nd}$.

(Either m or n may be unity; that is, the terms of either ratio may be multiplied without multiplying the terms of the other.)

341. *In any proportion, if the first and third terms be multiplied by any number, as also the second and fourth terms, the resulting numbers will be in proportion.*

Let the proportion be $\frac{a}{b}=\frac{c}{d}$; then, $\frac{ma}{nb}=\frac{mc}{nd}$.

(Either m or n may be unity.)

342. *In any number of proportions, the products of the corresponding terms are in proportion.*

Let the proportions be $\frac{a}{b}=\frac{c}{d}$, and $\frac{e}{f}=\frac{g}{h}$.

Multiplying, $$\frac{a}{b}\times\frac{e}{f}=\frac{c}{d}\times\frac{g}{h}, \text{ or } \frac{ae}{bf}=\frac{cg}{dh}.$$

In like manner, the theorem may be proved for any number of proportions.

343. *In any proportion, like powers or like roots of the terms are in proportion.*

Let the proportion be $\frac{a}{b}=\frac{c}{d}$; then, $\frac{a^n}{b^n}=\frac{c^n}{d^n}$.

In like manner, $\frac{\sqrt[n]{a}}{\sqrt[n]{b}}=\frac{\sqrt[n]{c}}{\sqrt[n]{d}}$.

344. *In a series of equal ratios, any antecedent is to its consequent as the sum of all the antecedents is to the sum of all the consequents.*

Let $a:b=c:d=e:f.$

Then by § 331, $ad=bc,$

and $af=be.$

Also, $ab=ba.$

Adding, $a(b+d+f)=b(a+c+e).$

Whence, $a:b=a+c+e:b+d+f.$ (§ 333)

In like manner, the theorem may be proved for any number of equal ratios.

345. *If three numbers are in continued proportion, the first is to the third as the square of the first is to the square of the second.*

Let the proportion be $a:b=b:c$; or $\frac{a}{b}=\frac{b}{c}$.

Then, $\frac{a}{b}\times\frac{b}{c}=\frac{a}{b}\times\frac{a}{b}$, or $\frac{a}{c}=\frac{a^2}{b^2}$.

346. *If four numbers are in continued proportion, the first is to the fourth as the cube of the first is to the cube of the second.*

Let the proportion be $a:b=b:c=c:d$; or $\frac{a}{b}=\frac{b}{c}=\frac{c}{d}$.

Then, $\frac{a}{b}\times\frac{b}{c}\times\frac{c}{d}=\frac{a}{b}\times\frac{a}{b}\times\frac{a}{b}$, or $\frac{a}{d}=\frac{a^3}{b^3}$.

347. Examples.

1. If $x : y = (x+z)^2 : (y+z)^2$, prove z the mean proportional between x and y.

From the given proportion, by § 331,

$$y(x+z)^2 = x(y+z)^2.$$

Or $$x^2y + 2\,xyz + yz^2 = xy^2 + 2\,xyz + xz^2.$$

Transposing, $$x^2y - xy^2 = xz^2 - yz^2.$$

Dividing by $x - y$, $$xy = z^2.$$

Therefore, z is the mean proportional between x and y (§ 336).

The theorem of § 339 saves work in the solution of a certain class of fractional equations.

2. Solve the equation $\dfrac{2\,x+3}{2\,x-3} = \dfrac{2\,b-a}{2\,b+a}$.

Regarding this as a proportion, we have by composition and division,

$$\frac{4\,x}{6} = \frac{4\,b}{-2\,a}, \text{ or } \frac{2\,x}{3} = -\frac{2\,b}{a}; \text{ whence, } x = -\frac{3\,b}{a}.$$

3. Prove that if $\dfrac{a}{b} = \dfrac{c}{d}$, then

$$a^2 - b^2 : a^2 - 3\,ab = c^2 - d^2 : c^2 - 3\,cd.$$

Let $\dfrac{a}{b} = \dfrac{c}{d} = x$, whence, $a = bx$; then,

$$\frac{a^2 - b^2}{a^2 - 3\,ab} = \frac{b^2x^2 - b^2}{b^2x^2 - 3\,b^2x} = \frac{x^2 - 1}{x^2 - 3\,x} = \frac{\dfrac{c^2}{d^2} - 1}{\dfrac{c^2}{d^2} - \dfrac{3\,c}{d}} = \frac{c^2 - d^2}{c^2 - 3\,cd}.$$

Then, $$a^2 - b^2 : a^2 - 3\,ab = c^2 - d^2 : c^2 - 3\,cd.$$

EXERCISE 151

1. Find the mean proportional between 18 and 32.

2. Find the third term of a proportion whose first, second, and fourth terms are 24, 32, and 20, respectively.

3. Find the third proportional to $\frac{14}{9}$ and $\frac{7}{12}$.

4. Find the mean proportional between $1\frac{1}{4}$ and $24\frac{8}{9}$.

5. Find the fourth proportional to $4\frac{4}{9}$, $5\frac{5}{6}$, and $1\frac{5}{7}$.

6. Find the third proportional to a^3+8 and $a+2$.

7. Find the mean proportional between

$$\frac{x^2-x-12}{x-5} \text{ and } \frac{x^2-9x+20}{x+3}.$$

Solve the following equations:

8. $\dfrac{3x-8}{3x+4}=\dfrac{2x-5}{2x+7}$.

9. $\dfrac{4x+7}{4x-7}=\dfrac{7a+1}{5a-3}$.

10. $\dfrac{x^2+2x-3}{x^2-2x-3}=\dfrac{3x+2}{3x-2}$.

11. $\dfrac{x^2+\sqrt{3x-1}}{x^2-\sqrt{3x-1}}=\dfrac{x^2-\sqrt{2x+1}}{x^2+\sqrt{2x+1}}$.

12. $\begin{cases} \dfrac{x+y}{x-y}=\dfrac{a-b}{a+b}. \\ \dfrac{x-a^3}{x+a^3}=\dfrac{b^3+y}{b^3-y}. \end{cases}$

13. Find two numbers in the ratio 4 to 3, such that the difference of their squares shall be 112.

14. Find two numbers such that, if 9 be added to the first, and 7 subtracted from the second, they will be in the ratio $9:2$; while if 9 be subtracted from the first, and 7 added to the second, they will be in the ratio $9:11$.

15. Find two numbers in the ratio $a:b$, such that, if each be increased by c, they shall be in the ratio $m:n$.

16. Find three numbers in continued proportion whose sum is $\frac{57}{2}$, such that the quotient of the first by the second shall be $\frac{2}{3}$.

17. What number must be added to each of the numbers a, b, and c, so that the resulting numbers shall be in continued proportion?

18. Find a number such that, if it be subtracted from each term of the ratio $8:5$, the result is $\frac{20}{11}$ of what it would have been if the same number had been added to each term.

19. The second of three numbers is the mean proportional between the other two. The third number exceeds the sum of the other two by 20; and the sum of the first and third exceeds three times the second by 4. Find the numbers.

20. If $8a-5b:7a-4b=8b-5c:7b-4c$, prove c the third proportional to a and b.

21. If $ma+nb:pa-qb=mc+nd:pc-qd$, prove $a:b=c:d$.

22. If $x+y:y+z=\sqrt{x^2-y^2}:\sqrt{y^2-z^2}$, prove y the mean proportional between x and z.

23. Given $(2a^2+2ab)x+(a^2+2b^2)y=(a^2-b^2)x+(2a^2+b^2)y$, find the ratio of x to y.

24. If 4 silver coins and 11 copper coins are worth as much as 2 gold coins, and 5 silver coins and 19 copper coins as much as 3 gold coins, find the ratio of the value of a gold coin, and the value of a silver coin, to the value of a copper coin.

If $\frac{a}{b}=\frac{c}{d}$, prove

25. $3a+4b:3a-4b=3c+4d:3c-4d$.

26. $a^2-5ab:2ab+7b^2=c^2-5cd:2cd+7d^2$.

27. $a^3+6ab^2:a^2b-8b^3=c^3+6cd^2:c^2d-8d^3$.

28. Each of two vessels contains a mixture of wine and water. A mixture consisting of equal measures from the two vessels is composed of wine and water in the ratio $3:4$; another mixture consisting of 2 measures from the first and 3 measures from the second, is composed of wine and water in the ratio $2:3$. Find the ratio of wine to water in each vessel.

XXVI. VARIATION

348. One variable number (§ 317) is said to *vary directly* as another when the ratio of any two values of the first equals the ratio of the corresponding values of the second.

It is usual to omit the word "directly," and simply say that one number *varies* as another.

Thus, if a workman receives a fixed number of dollars per diem, the number of dollars received in m days will be to the number received in n days as m is to n.

Then, the ratio of any two numbers of dollars received equals the ratio of the corresponding numbers of days worked.

Hence, the number of dollars which the workman receives *varies* as the number of days during which he works.

349. The symbol $\propto$ is read "*varies as*"; thus, $a \propto b$ is read "a varies as b."

350. One variable number is said to *vary inversely* as another when the first varies directly as the *reciprocal* of the second.

Thus, the number of hours in which a railway train will traverse a fixed route varies inversely as the speed; if the speed be *doubled,* the train will traverse its route in *one-half* the number of hours.

351. One variable number is said to vary as two others *jointly* when it varies directly as their product.

Thus, the number of dollars received by a workman in a certain number of days varies jointly as the number which he receives in one day, and the number of days during which he works.

352. One variable number is said to vary directly as a second and inversely as a third, when it varies jointly as the second and the reciprocal of the third.

Thus, the attraction of a body varies directly as the amount of matter, and inversely as the square of the distance.

353. *If $x \propto y$, then x equals y multiplied by a constant number.*

Let x' and y' denote a *fixed* pair of corresponding values of x and y, and x and y any other pair.

By the definition of § 348, $\frac{x}{y} = \frac{x'}{y'}$; or, $x = \frac{x'}{y'}y$.

Denoting the constant ratio $\frac{x'}{y'}$ by m, we have

$$x = my.$$

354. It follows from §§ 350, 351, 352, and 353 that:

1. *If x varies inversely as y,* $x = \frac{m}{y}$.
2. *If x varies jointly as y and z,* $x = myz$.
3. *If x varies directly as y and inversely as z,* $x = \frac{my}{z}$.

355. *If $x \propto y$, and $y \propto z$, then $x \propto z$.*

By § 353, if $x \propto y$, $\quad x = my. \quad (1)$

And if $y \propto z$, $\quad y = nz.$

Substituting in (1), $\quad x = mnz.$

Whence, by § 353, $\quad x \propto z.$

356. *If $x \propto y$ when z is constant, and $x \propto z$ when y is constant, then $x \propto yz$ when both y and z vary.*

Let y' and z' be the values of y and z, respectively, when x has the value x'.

Let y be changed from y' to y'', z remaining constantly equal to z', and let x be changed in consequence from x' to X.

Then by § 348, $\quad \frac{x'}{X} = \frac{y'}{y''}. \quad (1)$

Now let z be changed from z' to z'', y remaining constantly equal to y'', and let x be changed in consequence from X to x''.

Then, $$\frac{X}{x''}=\frac{z'}{z''}. \qquad (2)$$

Multiplying (1) by (2), $$\frac{x'}{x''}=\frac{y'z'}{y''z''}. \qquad (3)$$

Now if *both* changes are made, that is, y from y' to y'' and z from z' to z'', x is changed from x' to x'', and yz is changed from $y'z'$ to $y''z''$.

Then by (3), the ratio of any two values of x equals the ratio of the *corresponding values* of yz; and, by § 348, $x \propto yz$.

The following is an illustration of the above theorem:

It is known, by Geometry, that the area of a triangle varies as the base when the altitude is constant, and as the altitude when the base is constant; hence, when both base and altitude vary, the area varies as their product.

357. Problems.

Problems in variation are readily solved by converting the variation into an equation by aid of §§ 353 or 354.

1. If x varies inversely as y, and equals 9 when $y=8$, find the value of x when $y=18$.

If x varies inversely as y, $x=\frac{m}{y}$ (§ 354).

Putting $x=9$ and $y=8$, $9=\frac{m}{8}$, or $m=72$.

Then, $x=\frac{72}{y}$; and, if $y=18$, $x=\frac{72}{18}=4$.

2. Given that the area of a triangle varies jointly as its base and altitude, what will be the base of a triangle whose altitude is 12, equivalent to the sum of two triangles whose bases are 10 and 6, and altitudes 3 and 9, respectively?

Let B, H, and A denote the base, altitude, and area, respectively, of any triangle, and B' the base of the required triangle.

Since A varies jointly as B and H, $A=mBH$ (§ 354).

Therefore, the area of the first triangle is $m\times 10\times 3$, or $30\,m$, and the area of the second is $m\times 6\times 9$, or $54\,m$.

Then, the area of the required triangle is $30\,m+54\,m$, or $84\,m$.

But, the area of the required triangle is also $m \times B' \times 12$.

Therefore, $\quad 12\, mB' = 84\, m$, or $B' = 7$.

EXERCISE 152

1. If $y \propto x$, and x equals 6 when y equals 54, what is the value of y when x equals 8?

2. If x varies inversely as y, and equals $\frac{2}{3}$ when $y = \frac{3}{4}$, what is the value of y when $x = \frac{3}{2}$?

3. If $y \propto z^2$, and equals 40 when $z = 10$, what is the value of y in terms of z^2?

4. If z varies jointly as x and y, and equals $\frac{2}{3}$ when $x = \frac{3}{4}$ and $y = \frac{4}{5}$, what is the value of z when $x = \frac{4}{3}$ and $y = \frac{5}{4}$?

5. If x varies directly as y and inversely as z, and equals $\frac{9}{16}$ when $y = 27$ and $z = 64$, what is the value of x when $y = 9$ and $z = 32$?

6. If $x^4 \propto y^3$, and $x = 4$ when $y = 4$, what is the value of y when $x = \frac{1}{2}$?

7. If $5x + 8 \propto 6y - 1$, and $x = 6$ when $y = -3$, what is the value of x when $y = 7$?

8. The surface of a cube varies as the square of its edge. If the surface of a cube whose edge is $\frac{7}{3}$ feet is $\frac{98}{3}$ square feet, what will be the edge of a cube whose surface is $\frac{242}{3}$ square feet?

9. If 5 men in 6 days earn \$57, how many days will it take 4 men to earn \$76; it being given that the amount earned varies jointly as the number of men, and the number of days during which they work.

10. The volume of a sphere varies jointly as its diameter and surface. If the volume of a sphere whose diameter is a, and surface b, is c, what is the diameter of a sphere whose surface is p and volume q?

11. The distance fallen by a body from rest varies as the square of the time during which it falls. If it falls 579 feet in 6 seconds, how long will it take to fall $402\frac{1}{12}$ feet?

12. A circular plate of lead, 17 inches in diameter, is melted and formed into three circular plates of the same thickness. If the diameters of two of the plates are 8 and 9 inches respectively, find the diameter of the other; it being given that the area of a circle varies as the square of its diameter.

13. If y equals the sum of two numbers which vary directly as x^3 and inversely as x, respectively, and y equals -53 when x equals -3, and $\frac{29}{2}$ when x equals 2, what is the value of y when x equals $\frac{1}{2}$?

14. If x equals the sum of two numbers, one of which varies directly as y^2 and the other inversely as z^2, and $x = 45$ when $y = 1$ and $z = 1$, and $x = 40$ when $y = 2$ and $z = 3$, find the value of y when $x = 37$ and $z = 1$.

15. If y equals the sum of three numbers, the first of which is constant, and the second and third vary as x^2 and x^3, respectively, and $y = -50$ when $x = 2$, 30 when $x = -2$, and 110 when $x = -3$, find the expression for y in terms of x.

16. The volume of a circular coin varies jointly as its thickness and the square of the radius of its face. Two coins whose thicknesses are 5 and 7 units, and radii of faces 60 and 30 units, respectively, are melted and formed into 100 coins, each 3 units thick. Find the radius of the face of the new coin.

17. The weight of a spherical shell, 2 inches thick, is $\frac{19}{27}$ of its weight if solid. Find its diameter, it being given that the volume of a sphere varies as the cube of its diameter.

PROBLEMS IN PHYSICS

1. When the force which stretches a spring, a straight wire, or any elastic body is varied, it is found that the displacement produced in the body is always directly proportional to the force which acts upon it; *i.e.*, if d_1 and d_2 represent any two displacements, and f_1 and f_2 respectively the forces which produce them, then the algebraic statement of the above law is

$$\frac{d_1}{d_2} = \frac{f_1}{f_2}. \tag{1}$$

If a force of 2 pounds stretches a given wire .01 inch, how much will a force of 20 pounds stretch the same wire?

2. If the same force is applied to two wires of the same length and material, but of different diameters, D_1 and D_2, then the displacements d_1 and d_2 are found to be inversely proportional to the squares of the diameters, *i.e.*,

$$\frac{d_1}{d_2}=\frac{D_2^2}{D_1^2}. \tag{2}$$

If a weight of 100 kilograms stretches a wire .5 millimeter in diameter through 1 millimeter, how much elongation will the same weight produce in a wire 1.5 millimeters in diameter?

3. If the same force is applied to two wires of the same diameter and material, but of different lengths, l_1 and l_2, then it is found that

$$\frac{d_1}{d_2}=\frac{l_1}{l_2}. \tag{3}$$

From (1), (2), and (3) and § 356, it follows that when lengths, diameters, and forces are all different,

$$\frac{d_1}{d_2}=\frac{f_1}{f_2}\times\frac{l_1}{l_2}\times\frac{D_2^2}{D_1^2}. \tag{4}$$

If a force of 1 pound will stretch an iron wire which is 200 centimeters long and .5 millimeter in diameter through 1 millimeter, what force is required to stretch an iron wire 150 centimeters long and 1.25 millimeters in diameter through .5 millimeter?

4. When the temperature of a gas is constant, its volume is found to be inversely proportional to the pressure to which the gas is subjected, *i.e.*, algebraically stated,

$$\frac{V_1}{V_2}=\frac{P_2}{P_1}. \tag{5}$$

At the bottom of a lake 30 meters deep, where the pressure is 4000 grams per square centimeter, a bubble of air has a volume of 1 cubic centimeter as it escapes from a diver's suit. To what volume will it have expanded when it reaches the surface where the atmospheric pressure is about 1000 grams per square centimeter?

5. The electrical resistance of a wire varies directly as its length and inversely as its area. If a copper wire 1 centimeter in diameter has a resistance of 1 unit per mile, how many units of resistance will a copper wire have which is 500 feet long and 3 millimeters in diameter?

6. The illumination from a source of light varies inversely as the square of the distance from the source. A book which is now 10 inches from the source is moved 15 inches farther away. How much will the light received be reduced?

7. The period of vibration of a pendulum is found to vary directly as the square root of its length. If a pendulum 1 meter long ticks seconds, what will be the period of vibration of a pendulum 30 centimeters long?

8. The force with which the earth pulls on any body outside of its surface is found to vary inversely as the square of the distance from its center. If the surface of the earth is 4000 miles from the center, what would a pound weight weigh 15,000 miles from the earth?

9. The number of vibrations made per second by a guitar string of given diameter and material is inversely proportional to its length and directly proportional to the square root of the force with which it is stretched. If a string 3 feet long, stretched with a force of 20 pounds, vibrates 400 times per second, find the number of vibrations made by a string 1 foot long, stretched by a force of 40 pounds.

GRAPHS IN PHYSICS

1. *Graphical representation of a direct proportion.*

When a man is running at a constant speed, the distance which he travels in a given time is directly proportional to his speed. The algebraic expression of this relation is $\frac{d_1}{d_2} = \frac{s_1}{s_2}$, or $d = ms$. (See § 353.)

Now, if we plot successive values of the distance, d, which correspond to various speeds, s, in precisely the same manner in which we plotted successive values of x and y in § 181, we obtain as the graphical picture of the relation between s and d a straight line passing through the origin. (See Fig. 1.)

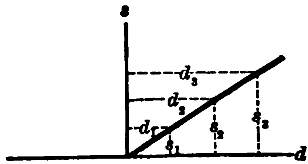


FIG. 1.

This is the graph of any direct proportion.

2. *Graphical representation of an inverse proportion.*

The volume which a given body of gas occupies when the pressure to which it is subjected varies has been found to be inversely proportional to the pressure under which the gas stands; we have seen that the algebraic statement of this relation is $\frac{V_1}{V_2} = \frac{P_2}{P_1}$.

If we plot successive values of V and P in the manner indicated in § 181, we obtain a graph of the form shown in Fig. 2.

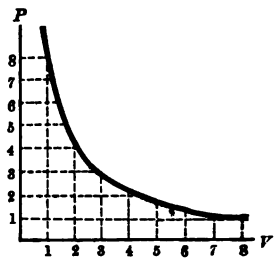


FIG. 2.

$$\frac{V_1}{V_2} = \frac{P_2}{P_1}, \text{ or } V = \frac{m}{P}.$$

$V = 1,\quad P = m.\qquad V = 5,\quad P = \frac{m}{5}.$

$V = 2,\quad P = \frac{m}{2}.\qquad V = 6,\quad P = \frac{m}{6}.$

$V = 3,\quad P = \frac{m}{3}.\qquad V = 7,\quad P = \frac{m}{7}.$

$V = 4,\quad P = \frac{m}{4}.\qquad V = 8,\quad P = \frac{m}{8}.$

This is the *graphical representation of any inverse proportion;* the curve is called an **equilateral hyperbola.**

3. *The path traversed by a falling body projected horizontally.*

When a body is thrown horizontally from the top of a tower, if it were not for gravity, it would move on in a horizontal direction indefinitely, traversing exactly the same distance in each succeeding second.

Hence, if V represents the velocity of projection, the horizontal distance, H, which it would traverse in any number of seconds, t, would be given by the equation $H = Vt$.

On account of gravity, however, the body is pulled downward, and traverses in this direction in any number of seconds a distance which is given by the equation $S = \frac{1}{2}gt^2$.

To find the actual path taken by the body, we have only to plot successive values of H and S, in the manner in which we plotted the successive values of x and y, in § 181.

Thus, at the end of 1 second the vertical distance S_1 is given by $S_1 = \frac{1}{2}g \times 1^2 = \frac{1}{2}g$; at the end of 2 seconds we have $S_2 = \frac{1}{2}g \times 2^2 = \frac{4}{2}g$; at the end of 3 seconds, $S_3 = \frac{1}{2}g \times 3^2 = \frac{9}{2}g$; at the end of 4 seconds, $S_4 = \frac{1}{2}g \times 4^2 = \frac{16}{2}g$; etc.

On the other hand, at the end of 1 second we have $H_1 = V$; at the end of 2 seconds, $H_2 = 2V$; at the end of 3 seconds, $H_3 = 3V$; at the end of 4 seconds, $H_4 = 4V$.

If, now, we plot these successive values of H and S, we obtain the graph shown in Fig. 3.

This is the path of the body; it is a parabola. (§ 314, Ex. 2.)

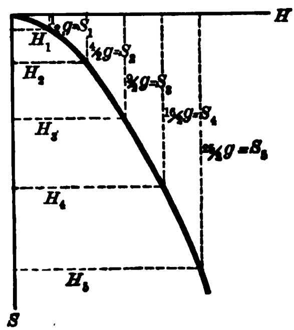


FIG. 3.

4. *Graph of relation between the temperature and pressure existing within an air-tight boiler containing only water and water vapor.*

One use of graphs in physics is to express a relation which is found by experiment to exist between two quantities, which cannot be represented by any simple algebraic equation.

For example, when the temperature of an air-tight boiler which contains only water and water vapor is raised, the pressure within the boiler increases also; thus we find by direct experiment that when the temperature of the boiler is 0° centigrade, the pressure which the vapor exerts will support a column of mercury 4.6 millimeters high.

When the temperature is raised to 10°, the mercury column rises to 9.1 millimeters; at 30° the column is 31.5 millimeters long, etc.

To obtain a simple and compact picture of the relation between temperature and pressure, we plot a succession of temperatures, *e.g.* 0°, 10°, 20°, 30°, 40°, 50°, 60°, 70°, 80°, 90°, 100°, in the manner in which we plotted successive values of x in § 181, and then plot the corresponding values of pressure obtained by experiment in the manner in which we plotted the y's in § 181; we obtain the graph shown in Fig. 4.

From this graph we can find at once the pressure which will exist within the boiler at any temperature.

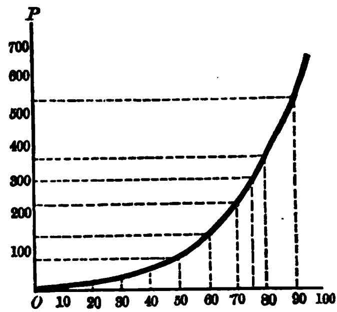


FIG. 4.

For example, if we wish to know the pressure at 75° centigrade, we observe where the vertical line which passes through 75° cuts the curve and then run a horizontal line from this point to the point of intersection with the line OP.

This point is found to be at 288; hence the pressure within the boiler at 75° centigrade is 288 millimeters.

XXVII. PROGRESSIONS

ARITHMETIC PROGRESSION

358. An **Arithmetic Progression** is a series of terms in which each term, after the first, is obtained by adding to the preceding term a constant number called the *Common Difference.*

Thus, 1, 3, 5, 7, 9, 11, $\cdots$ is an arithmetic progression in which the common difference is 2.

Again, 12, 9, 6, 3, 0, -3, $\cdots$ is an arithmetic progression in which the common difference is -3.

An Arithmetic Progression is also called an *Arithmetic Series.*

359. *Given the first term, a, the common difference, d, and the number of terms, n, to find the last term, l.*

The progression is a, $a+d$, $a+2d$, $a+3d$, $\cdots$.

We observe that the coefficient of d in any term is less by 1 than the number of the term.

Then, in the nth term the coefficient of d will be $n-1$.

That is, $$l=a+(n-1)d. \qquad \text{(I)}$$

360. *Given the first term, a, the last term, l, and the number of terms, n, to find the sum of the terms, S.*

$$S=a+(a+d)+(a+2d)+\cdots+(l-d)+l.$$

Writing the terms in reverse order,

$$S=l+(l-d)+(l-2d)+\cdots+(a+d)+a.$$

Adding these equations term by term,

$$2S=(a+l)+(a+l)+(a+l)+\cdots+(a+l)+(a+l).$$

Therefore, $2S=n(a+l)$, and $S=\frac{n}{2}(a+l)$. (II)

361. Substituting in (II) the value of l from (I), we have

$$S=\frac{n}{2}[2a+(n-1)d].$$

362. *Ex.* In the progression 8, 5, 2, -1, -4, $\cdots$, to 27 terms, find the last term and the sum.

Here, $a = 8,\ d = 5 - 8 = -3,\ n = 27.$

Substitute in (I), $l = 8 + (27 - 1)(-3) = 8 - 78 = -70.$

Substitute in (II), $S = \frac{27}{2}(8 - 70) = 27(-31) = -837.$

The common difference may be found by subtracting the first term from the second, or any term from the next following term.

EXERCISE 153

In each of the following, find the last term and the sum:

1. $4, 9, 14, \cdots$ to 14 terms.
2. $9, 2, -5, \cdots$ to 16 terms.
3. $-51, -45, -39, \cdots$ to 15 terms.
4. $-\frac{7}{4}, -\frac{19}{8}, -3, \cdots$ to 13 terms.
5. $\frac{5}{6}, \frac{1}{6}, -\frac{1}{2}, \cdots$ to 18 terms.
6. $\frac{8}{9}, \frac{47}{36}, \frac{31}{18}, \cdots$ to 17 terms.
7. $-\frac{11}{8}, -\frac{19}{16}, -1, \cdots$ to 27 terms.
8. $-\frac{3}{10}, -\frac{5}{6}, -\frac{41}{30}, \cdots$ to 52 terms.
9. $3a + 4b, 8a + 2b, 13a, \cdots$ to 10 terms.
10. $\frac{x - 2y}{3}, \frac{x}{6}, \frac{2y}{3}, \cdots$ to 9 terms.

363. The *first term, common difference, number of terms, last term,* and *sum of the terms,* are called the *elements* of the progression.

If any three of the five elements of an arithmetic progression are given, the other two may be found by substituting the known values in the fundamental formulæ (I) and (II), and solving the resulting equations.

1. Given $a = -\frac{5}{3}$, $n = 20$, $S = -\frac{5}{3}$; find d and l.

Substituting the given values in (II),

$$-\frac{5}{3} = 10\left(-\frac{5}{3} + l\right), \text{ or } -\frac{1}{6} = -\frac{5}{3} + l; \text{ then, } l = \frac{5}{3} - \frac{1}{6} = \frac{3}{2}.$$

Substituting the values of a, n, and l in (I), $\frac{3}{2} = -\frac{5}{3} + 19\,d.$

Whence, $$19\,d = \frac{3}{2} + \frac{5}{3} = \frac{19}{6}, \text{ and } d = \frac{1}{6}.$$

2. Given $d = -3$, $l = -39$, $S = -264$; find a and n.

Substituting in (I), $-39 = a + (n-1)(-3)$, or $a = 3\,n - 42.$ (1)

Substituting the values of l, S, and a in (II),

$$-264 = \frac{n}{2}(3\,n - 42 - 39), \text{ or } -528 = 3\,n^2 - 81\,n, \text{ or } n^2 - 27\,n + 176 = 0.$$

Whence, $$n = \frac{27 \pm \sqrt{729 - 704}}{2} = \frac{27 \pm 5}{2} = 16 \text{ or } 11.$$

Substituting in (1), $a = 48 - 42$ or $33 - 42 = 6$ or -9.

The solution is $a = 6$, $n = 16$; or, $a = -9$, $n = 11$.

The significance of the two answers is as follows:

If $a = 6$ and $n = 16$, the progression is $6, 3, 0, -3, -6, -9, -12, -15, -18, -21, -24, -27, -30, -33, -36, -39$.

If $a = -9$ and $n = 11$, the progression is

$$-9, -12, -15, -18, -21, -24, -27, -30, -33, -36, -39.$$

In each of these the sum is -264.

3. Given $a = \frac{1}{3}$, $d = -\frac{1}{12}$, $S = -\frac{3}{2}$; find l and n.

Substituting in (I), $l = \frac{1}{3} + (n-1)\left(-\frac{1}{12}\right) = \frac{5-n}{12}.$ (1)

Substituting the values of a, S, and l in (II),

$$-\frac{3}{2} = \frac{n}{2}\left(\frac{1}{3} + \frac{5-n}{12}\right), \text{ or } -3 = n\left(\frac{9-n}{12}\right), \text{ or } n^2 - 9\,n - 36 = 0.$$

Whence, $$n = \frac{9 \pm \sqrt{81 + 144}}{2} = \frac{9 \pm 15}{2} = 12 \text{ or } -3.$$

The value $n = -3$ must be rejected, for the number of terms in a progression must be a *positive integer*.

Substituting $n = 12$ in (1), $l = \frac{5-12}{12} = -\frac{7}{12}.$

A *negative* or *fractional* value of n must be rejected, together with all other values dependent on it.

EXERCISE 154

1. Given $d=8$, $l=115$, $n=15$; find a and S.
2. Given $d=-6$, $n=14$, $S=-616$; find a and l.
3. Given $a=-69$, $n=16$, $l=36$; find d and S.
4. Given $a=8$, $n=25$, $S=-2500$; find d and l.
5. Given $a=\frac{3}{4}$, $l=-\frac{51}{4}$, $S=-78$; find d and n.
6. Given $l=\frac{117}{4}$, $n=13$, $S=\frac{793}{4}$; find a and d.
7. Given $a=-\frac{6}{5}$, $d=-\frac{3}{10}$, $S=-\frac{306}{5}$; find n and l.
8. Given $a=-\frac{9}{2}$, $l=\frac{64}{3}$, $d=\frac{5}{6}$; find n and S.
9. Given $d=-\frac{1}{12}$, $n=55$, $S=-165$; find a and l.
10. Given $l=\frac{227}{12}$, $n=24$, $S=241$; find a and d.
11. Given $l=\frac{95}{6}$, $d=\frac{5}{6}$, $S=\frac{935}{6}$; find a and n.
12. Given $a=-\frac{4}{5}$, $l=-\frac{97}{10}$, $S=-\frac{315}{2}$; find d and n.
13. Given $a=-\frac{9}{22}$, $n=21$, $S=\frac{21}{22}$; find d and l.
14. Given $l=\frac{23}{12}$, $d=\frac{5}{12}$, $S=-\frac{53}{8}$; find a and n.
15. Given $a=-\frac{31}{6}$, $d=\frac{1}{3}$, $S=-\frac{77}{2}$; find n and l.

364. From (I) and (II), *general formulæ* for the solution of examples like the above may be readily derived.

Ex. Given a, d, and S; derive the formula for n.

By § 361, $2S=n[2a+(n-1)d]$, or $dn^2+(2a-d)n=2S$.

This is a quadratic in n, and may be solved by the method of § 288; multiplying by $4d$, and adding $(2a-d)^2$ to both members,

$$4d^2n^2+4d(2a-d)n+(2a-d)^2=8dS+(2a-d)^2.$$

Extracting square roots, $2dn+2a-d=\pm\sqrt{8dS+(2a-d)^2}$.

Whence, $$n=\frac{d-2a\pm\sqrt{8dS+(2a-d)^2}}{2d}.$$

EXERCISE 155

1. Given a, l, and n; derive the formula for d.

2. Given a, n, and S; derive the formulæ for d and l.

3. Given d, n, and S; derive the formulæ for a and l.

4. Given a, d, and l; derive the formulæ for n and S.

5. Given d, l, and n; derive the formulæ for a and S.

6. Given l, n, and S; derive the formulæ for a and d.

7. Given a, d, and S; derive the formula for l.

8. Given a, l, and S; derive the formulæ for d and n.

9. Given d, l, and S; derive the formulæ for a and n.

365. Arithmetic Means.

We define *inserting m arithmetic means between two given numbers, a and b,* as finding an arithmetic progression of $m+2$ terms, whose first and last terms are a and b.

Ex. Insert 5 arithmetic means between 3 and -5.

We find an arithmetic progression of 7 terms, in which $a=3$, and $l=-5$; substituting $n=7$, $a=3$, and $l=-5$ in (I),

$$-5=3+6d, \text{ or } d=-\frac{4}{3}.$$

The progression is $3,\ \frac{5}{3},\ \frac{1}{3},\ -1,\ -\frac{7}{3},\ -\frac{11}{3},\ -5.$

366. Let x denote the arithmetic mean between a and b.

Then, $\qquad x-a=b-x, \text{ or } 2x=a+b.$

Whence, $$x=\frac{a+b}{2}.$$

That is, *the arithmetic mean between two numbers equals one-half their sum.*

EXERCISE 156

1. Insert 7 arithmetic means between 4 and 10.

2. Insert 6 arithmetic means between $-\frac{5}{6}$ and $-\frac{11}{2}$.

3. Insert 9 arithmetic means between $-\frac{7}{8}$ and 6.

4. Insert 8 arithmetic means between -3 and $-\frac{33}{4}$.

5. Insert 5 arithmetic means between $\frac{5}{9}$ and $-\frac{1}{8}$.

6. How many arithmetic means are inserted between $-\frac{3}{2}$ and $\frac{29}{10}$, when the sum of the second and last is $\frac{9}{5}$?

7. If m arithmetic means are inserted between a and b, find the first two.

Find the arithmetic mean between:

8. $\frac{16}{9}$ and $-\frac{17}{6}$.

9. $(3m+n)^2$ and $(m-3n)^2$.

10. $\dfrac{x}{x-1}$ and $-\dfrac{x}{x^3-1}$.

367. Problems.

1. The sixth term of an arithmetic progression is $\frac{5}{6}$, and the fifteenth term is $\frac{16}{3}$. Find the first term.

By § 359, the sixth term is $a + 5d$, and the fifteenth term $a + 14d$.

Then by the conditions,
$$\begin{cases} a + 5d = \dfrac{5}{6}, & (1) \\ a + 14d = \dfrac{16}{3}. & (2) \end{cases}$$

Subtracting (1) from (2), $9d = \dfrac{9}{2}$; whence, $d = \dfrac{1}{2}$.

Substituting in (1), $a + \dfrac{5}{2} = \dfrac{5}{6}$; whence, $a = -\dfrac{5}{3}$.

2. Find four numbers in arithmetic progression such that the product of the first and fourth shall be 45, and the product of the second and third 77.

Let the numbers be $x - 3y$, $x - y$, $x + y$, and $x + 3y$.

Then by the conditions,
$$\begin{cases} x^2 - 9y^2 = 45. \\ x^2 - y^2 = 77. \end{cases}$$

Solving these equations, $x = 9$, $y = \pm 2$; or, $x = -9$, $y = \pm 2$ (§ 308).
Then the numbers are 3, 7, 11, 15; or, -3, -7, -11, -15.

In problems like the above, it is convenient to represent the unknown numbers by *symmetrical* expressions.

Thus, if five numbers had been required, we should have represented them by $x - 2y$, $x - y$, x, $x + y$, and $x + 2y$.

EXERCISE 157

1. The fifth term of an arithmetic progression is $\frac{13}{6}$, and the thirteenth term $\frac{7}{2}$. Find the twenty-second term.

2. Find the sum of all the odd integers, beginning with 1 and ending with 999.

3. How many positive integers of three digits are multiples of 7? What is their sum?

4. The first term of an arithmetic progression is 1, and the sum of the sixth and tenth terms is 37. Find the second and third terms.

5. The first term of an arithmetic progression of 11 terms is $\frac{3}{4}$, and the seventh term -3. Find the sum of the terms.

6. In an arithmetic progression, the sum of the first and last terms is two-ninths the sum of all the terms. Find the number of terms.

7. The seventh term of an arithmetic progression is -37, and the sum of the first 17 terms -799. Find the sum of the first 13 terms.

8. Find five numbers in arithmetic progression such that the sum of the first, fourth, and fifth is 14, and the quotient of the second by the fourth $-\frac{1}{4}$.

9. How many arithmetic means are inserted between $-\frac{3}{2}$ and $\frac{9}{2}$, when their sum is $\frac{21}{2}$?

10. If the constant difference of an arithmetic progression equals twice the first term, the quotient of the sum of the terms by the first term equals the square of the number of terms.

11. The sum of the first 10 terms of an arithmetic progression is to the sum of the first 5 terms as 13 to 4. Find the ratio of the first term to the common difference.

12. Find four numbers in arithmetic progression such that the sum of the first and second shall be -1, and the product of the second and fourth 24.

13. The last term of an arithmetic progression of 10 terms is 29. The sum of the odd-numbered terms is to the sum of the even-numbered terms as 14 is to 17. Find the first term and the common difference.

14. The sum of five numbers in arithmetic progression is 25, and the sum of their squares is 135. Find the numbers.

15. A man travels $\frac{891}{2}$ miles. He travels 10 miles the first day, and increases his speed one-half mile in each succeeding day. How many days does the journey require?

16. Find the sum of the terms of an arithmetic progression of 9 terms, in which 17 is the middle term.

17. Find three numbers in arithmetic progression, such that the square of the first added to the product of the other two gives 16, and the square of the second added to the product of the other two gives 14.

18. If a person saves \$120 each year, and puts this sum at simple interest at $3\frac{1}{2}\%$ at the end of each year, to how much will his property amount at the end of 18 years?

19. A traveller sets out from a certain place, and goes 7 miles the first hour, $7\frac{1}{2}$ the second hour, 8 the third hour, and so on. After he has been gone 5 hours, another sets out, and travels $16\frac{1}{4}$ miles an hour. How many hours after the first starts are the travellers together?

20. There are 12 equidistant balls in a straight line. A person starts from a position in line with the balls, and beyond them, his distance from the first ball being the same as the distance between the balls, and picks them up in succession, returning with each to his original position. He finds that he has walked 780 feet. Find the distance between the balls.

GEOMETRIC PROGRESSION

368. A **Geometric Progression** is a series of terms in which each term, after the first, is obtained by multiplying the preceding term by a constant number called the *Ratio*.

Thus, 2, 6, 18, 54, 162, $\cdots$ is a geometric progression in which the ratio is 3.

9, 3, 1, $\frac{1}{3}$, $\frac{1}{9}$, $\cdots$ is a geometric progression in which the ratio is $\frac{1}{3}$.

-3, 6, -12, 24, -48, $\cdots$ is a geometric progression in which the ratio is -2.

A Geometric Progression is also called a *Geometric Series.*

369. *Given the first term, a, the ratio, r, and the number of terms, n, to find the last term, l.*

The progression is $a, ar, ar^2, ar^3, \cdots$.

We observe that the exponent of r in any term is less by 1 than the number of the term.

Then, in the nth term the exponent of r will be $n-1$.

That is, $$l = ar^{n-1}. \qquad \text{(I)}$$

370. *Given the first term, a, the last term, l, and the ratio, r, to find the sum of the terms, S.*

$$S = a + ar + ar^2 + \cdots + ar^{n-3} + ar^{n-2} + ar^{n-1}. \qquad (1)$$

Multiplying each term by r,

$$rS = ar + ar^2 + ar^3 + \cdots + ar^{n-2} + ar^{n-1} + ar^n. \qquad (2)$$

Subtracting (1) from (2), $rS - S = ar^n - a$, or $S = \dfrac{ar^n - a}{r-1}$.

But by (I), § 369, $$rl = ar^n.$$

Therefore, $$S = \frac{rl - a}{r - 1}. \qquad \text{(II)}$$

The *first term, ratio, number of terms, last term,* and *sum of the terms,* are called the *elements* of the progression.

371. Examples.

1. In the progression 3, 1, $\frac{1}{3}$, $\cdots$, to 7 terms, find the last term and the sum.

Here, $a = 3$, $r = \dfrac{1}{3}$, $n = 7$.

Substituting in (I), $\quad l = 3\left(\frac{1}{3}\right)^6 = \frac{1}{3^5} = \frac{1}{243}.$

Substituting in (II), $\quad S = \dfrac{\frac{1}{3} \times \frac{1}{243} - 3}{\frac{1}{3} - 1} = \dfrac{\frac{1}{729} - 3}{-\frac{2}{3}} = \dfrac{-\frac{2186}{729}}{-\frac{2}{3}} = \dfrac{1093}{243}.$

The ratio may be found by dividing the second term by the first, or any term by the next preceding term.

2. In the progression $-2, 6, -18, \cdots$, to 8 terms, find the last term and the sum.

Here, $a = -2$, $r = \dfrac{6}{-2} = -3$, $n = 8$; therefore,

$$l = -2(-3)^7 = -2 \times (-2187) = 4374.$$

$$S = \frac{-3 \times 4374 - (-2)}{-3 - 1} = \frac{-13122 + 2}{-4} = 3280.$$

EXERCISE 158

Find the last term and the sum of the following:

1. $1, -2, 4, \cdots$ to 10 terms.
2. $-6, -9, -\frac{27}{2}, \cdots$ to 7 terms.
3. $3, -15, 75, \cdots$ to 5 terms.
4. $-5, -20, -80, \cdots$ to 6 terms.
5. $\frac{1}{4}, \frac{1}{2}, 1, \cdots$ to 9 terms.
6. $-\frac{3}{2}, \frac{1}{2}, -\frac{1}{6}, \cdots$ to 7 terms.
7. $-4, -3, -\frac{9}{4}, \cdots$ to 5 terms.
8. $-\frac{5}{6}, \frac{5}{2}, -\frac{15}{2}, \cdots$ to 8 terms.
9. $2, \frac{4}{5}, \frac{8}{25}, \cdots$ to 6 terms.
10. $\frac{3}{4}, -\frac{1}{2}, \frac{1}{3}, \cdots$ to 8 terms.

372. If any three of the five elements of a geometric progression are given, the other two may be found by substituting the given values in the fundamental formulæ (I) and (II), and solving the resulting equations.

But in certain cases the operation involves the solution of an equation of a degree higher than the second; and in others the unknown number appears as an exponent, the solution of which form of equation can usually only be effected by the aid of logarithms (§ 437).

In all such cases in the present chapter, the equations may be solved by inspection.

1. Given $a = -2$, $n = 5$, $l = -32$; find r and S.

Substituting the given values in (I), we have

$$-32 = -2r^4;\ \text{whence, } r^4 = 16, \text{ or } r = \pm 2.$$

Substituting in (II),

If $\quad r = 2,\ S = \dfrac{2(-32) - (-2)}{2 - 1} = -64 + 2 = -62.$

If $\quad r = -2,\ S = \dfrac{(-2)(-32) - (-2)}{-2 - 1} = \dfrac{64 + 2}{-3} = -22.$

The solution is $r = 2$, $S = -62$; or, $r = -2$, $S = -22$.

The interpretation of the two answers is as follows:

If $r = 2$, the progression is $-2, -4, -8, -16, -32$, whose sum is -62.

If $r = -2$, the progression is $-2, 4, -8, 16, -32$, whose sum is -22.

2. Given $a = 3$, $r = -\frac{1}{3}$, $S = \frac{1640}{729}$; find n and l.

Substituting in (II), $\quad \dfrac{1640}{729} = \dfrac{-\frac{1}{3}l - 3}{-\frac{1}{3} - 1} = \dfrac{l + 9}{4}.$

Whence, $\quad l + 9 = \dfrac{6560}{729};\ \text{or, } l = \dfrac{6560}{729} - 9 = -\dfrac{1}{729}.$

Substituting the values of a, r, and l in (I),

$$-\frac{1}{729} = 3\left(-\frac{1}{3}\right)^{n-1};\ \text{or, } \left(-\frac{1}{3}\right)^{n-1} = -\frac{1}{2187}.$$

Whence, by inspection, $n - 1 = 7$, or $n = 8$.

From (I) and (II) general formulæ may be derived for the solution of cases like the above.

If the given elements are n, l, and S, *equations* for a and r may be found, but there are no definite *formulæ* for their values.

The same is the case when the given elements are a, n, and S.

The general formulæ for n involve logarithms; these cases are discussed in § 437.

EXERCISE 159

1. Given $r = 3$, $n = 8$, $l = 2187$; find a and S.

2. Given $r = -4$, $n = 5$, $S = -410$; find a and l.

3. Given $a=6$, $n=6$, $l=-\frac{64}{81}$; find r and S.

4. Given $a=3$, $r=\frac{1}{2}$, $l=\frac{3}{128}$; find n and S.

5. Given $r=-2$, $n=10$, $S=-\frac{1705}{2}$; find a and l.

6. Given $a=\frac{9}{2}$, $n=7$, $l=\frac{32}{81}$; find r and S.

7. Given $a=-\frac{1}{3}$, $l=-\frac{243}{64}$, $S=-\frac{463}{192}$; find r and n.

8. Given $a=\frac{3}{4}$, $r=\frac{3}{4}$, $S=\frac{2343}{1024}$; find l and n.

9. Given $l=-768$, $r=4$, $S=-\frac{4095}{4}$; find a and n.

10. Given $a=\frac{2}{9}$, $l=1458$, $S=\frac{9842}{9}$; find r and n.

11. Given a, r, and S; derive the formula for l.

12. Given a, l, and S; derive the formula for r.

13. Given r, l, and S; derive the formula for a.

14. Given r, n, and l; derive the formulæ for a and S.

15. Given r, n, and S; derive the formulæ for a and l.

16. Given a, n, and l; derive the formulæ for r and S.

373. Sum of a Geometric Progression to Infinity.

The limit (§ 318) to which the sum of the terms of a *decreasing* geometric progression approaches, when the number of terms is indefinitely increased, is called the *sum of the series to infinity.*

Formula (II), § 370, may be written

$$S=\frac{a-rl}{1-r}.$$

It is evident that, by sufficiently continuing a decreasing geometric progression, the absolute value of the last term may be made less than any assigned number, however small.

Hence, when the number of terms is indefinitely increased, l, and therefore rl, approaches the limit 0.

Then, the fraction $\frac{a-rl}{1-r}$ approaches the limit $\frac{a}{1-r}$.

Therefore, the sum of a decreasing geometric progression to infinity is given by the formula

$$S = \frac{a}{1-r}. \qquad \text{(III)}$$

Ex. Find the sum of the series 4, $-\frac{8}{3}$, $\frac{16}{9}$, $\cdots$ to infinity.

Here, $a = 4$, $r = -\frac{2}{3}$.

Substituting in (III), $S = \dfrac{4}{1+\frac{2}{3}} = \dfrac{12}{5}$.

EXERCISE 160

Find the sum to infinity of the following:

1. $6, 2, \frac{2}{3}, \cdots$.
2. $12, -3, \frac{3}{4}, \cdots$.
3. $1, \frac{1}{6}, \frac{1}{36}, \cdots$.
4. $-\frac{25}{6}, \frac{25}{9}, -\frac{50}{27}, \cdots$.
5. $\frac{8}{5}, \frac{14}{15}, \frac{49}{90}, \cdots$.
6. $-\frac{1}{10}, \frac{2}{25}, -\frac{8}{125}, \cdots$.
7. $-\frac{3}{4}, -\frac{15}{32}, -\frac{75}{256}, \cdots$.
8. $\frac{5}{6}, -\frac{5}{27}, \frac{10}{243}, \cdots$.

374. *To find the value of a repeating decimal.*

This is a case of finding the sum of a decreasing geometric series to infinity, and may be solved by formula (III).

Ex. Find the value of .85151 $\cdots$.

We have, $.85151 \cdots = .8 + .051 + .00051 + \cdots$.

The terms after the first constitute a decreasing geometric progression, in which $a = .051$, and $r = .01$.

Substituting in (III), $S = \dfrac{.051}{1-.01} = \dfrac{.051}{.99} = \dfrac{51}{990} = \dfrac{17}{330}$.

Then, the value of the given decimal is $\dfrac{8}{10} + \dfrac{17}{330}$, or $\dfrac{281}{330}$.

EXERCISE 161

Find the values of the following:

1. .7272 $\cdots$.
2. .629629 $\cdots$.
3. .91777 $\cdots$.
4. .75959 $\cdots$.
5. .23135135 $\cdots$.
6. .587474 $\cdots$.

375. Geometric Means.

We define *inserting m geometric means between two numbers, a and b,* as finding a geometric progression of $m+2$ terms, whose first and last terms are a and b.

Ex. Insert 5 geometric means between 2 and $\frac{128}{729}$.

We find a geometric progression of 7 terms, in which $a=2$, and $l=\frac{128}{729}$; substituting $n=7$, $a=2$, and $l=\frac{128}{729}$ in (I),

$$\frac{128}{729}=2\,r^6;\text{ whence } r^6=\frac{64}{729},\text{ and } r=\pm\frac{2}{3}.$$

The result is $2,\ \pm\frac{4}{3},\ \frac{8}{9},\ \pm\frac{16}{27},\ \frac{32}{81},\ \pm\frac{64}{243},\ \frac{128}{729}$.

376. Let x denote the geometric mean between a and b.

Then, $\frac{x}{a}=\frac{b}{x}$, or $x^2=ab$.

Whence, $x=\sqrt{ab}$.

That is, *the geometric mean between two numbers is equal to the square root of their product.*

EXERCISE 162

1. Insert 4 geometric means between $\frac{3}{4}$ and 24.
2. Insert 5 geometric means between -3 and -2187.
3. Insert 4 geometric means between $\frac{5}{16}$ and -320.
4. Insert 6 geometric means between $-\frac{3}{8}$ and $\frac{16}{729}$.
5. Insert 7 geometric means between -48 and $-\frac{3}{16}$.
6. Insert 3 geometric means between $\frac{125}{32}$ and $\frac{1}{10}$.
7. If m geometric means are inserted between a and b, what are the last two means?

Find the geometric mean between:

8. $\frac{14}{3}$ and $\frac{21}{8}$.

9. $\frac{x^2+xy}{xy-y^2}$ and $\frac{x^2-y^2}{xy}$.

10. a^2-4a+4 and $4a^2+4a+1$.

377. Problem.

Find 3 numbers in geometric progression such that their sum shall be 14, and the sum of their squares 84.

Let the numbers be represented by a, ar, and ar^2.

Then, by the conditions, $\begin{cases} a + ar + ar^2 = 14. & (1) \\ a^2 + a^2r^2 + a^2r^4 = 84. & (2) \end{cases}$

Divide (2) by (1), $a - ar + ar^2 = 6. \quad (3)$

Subtract (3) from (1), $2\,ar = 8, \text{ or } r = \frac{4}{a}. \quad (4)$

Substituting in (1), $a + 4 + \frac{16}{a} = 14, \text{ or } a^2 - 10\,a + 16 = 0.$

Solving this equation, $a = 8 \text{ or } 2.$

Substituting in (4), $r = \frac{4}{8} \text{ or } \frac{4}{2} = \frac{1}{2} \text{ or } 2.$

Then, the numbers are 2, 4, and 8.

EXERCISE 163

1. The fifth term of a geometric progression is $\frac{2}{9}$, and the eighth term $-\frac{16}{243}$. Find the third term.

2. The product of the first five terms of a geometric progression is 243. Find the third term.

3. Find four numbers in geometric progression such that the sum of the first and fourth shall be 27, and of the second and third 18.

4. Find an arithmetic progression whose first term is 2, and whose first, fourth, and tenth terms form a geometric progression.

5. The third term of a geometric progression is $\frac{2}{3}$, and the seventh term $\frac{27}{128}$; find the ninth term.

6. The sum of the terms of a geometric progression whose first term is 1, ratio 3, and number of terms 4, equals the sum of the terms of an arithmetic progression whose first term is 4, and common difference 4. Find the number of terms of the arithmetic progression.

7. The sum of the first four terms of a decreasing geometric progression is to the sum to infinity as 16 to 25. Find the ratio.

8. A man who saved every year four-thirds as much as in the preceding year, had in four years saved $3500. How much did he save the first year?

9. The difference between two numbers is 16, and their arithmetic mean exceeds their geometric mean by 2. Find the numbers.

10. Find six numbers in geometric progression such that the sum of the first and fourth shall be 9, and of the third and sixth 36.

11. The digits of a number of three figures are in geometric progression, and their sum is 7. If 297 be added to the number, the digits will be reversed. Find the number.

12. There are three numbers in geometric progression whose sum is $\frac{19}{2}$. If the first be multiplied by $\frac{2}{3}$, the second by $\frac{2}{3}$, and the third by $\frac{16}{27}$, the resulting numbers will be in arithmetic progression. What are the numbers?

HARMONIC PROGRESSION

378. A **Harmonic Progression** is a series of terms whose reciprocals form an arithmetic progression.

Thus, $1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \cdots$ is a harmonic progression, because the reciprocals of the terms, $1, 3, 5, 7, 9, \cdots$, form an arithmetic progression.

A Harmonic Progression is also called a *Harmonic Series*.

379. Any problem in harmonic progression, which is susceptible of solution, may be solved by taking the reciprocals of the terms, and applying the formulæ of the arithmetic progression.

There is, however, no general method for finding the *sum of the terms* of a harmonic series.

Ex. In the progression $2, \frac{2}{3}, \frac{2}{5}, \cdots$ to 36 terms, find the last term.

Taking the reciprocals of the terms, we have the arithmetic progression

$$\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \cdots.$$

Here, $a=\frac{1}{2},\ d=1,\ n=36.$

Substituting in (I), § 359, $l=\frac{1}{2}+(36-1)\times 1=\frac{71}{2}.$

Then, $\frac{2}{71}$ is the last term of the given harmonic series.

380. Harmonic Means.

We define *inserting m harmonic means between two numbers, a and b,* as finding a harmonic progression of $m+2$ terms, whose first and last terms are a and b.

Ex. Insert 5 harmonic means between 2 and -3.

We have to insert 5 arithmetic means between $\frac{1}{2}$ and $-\frac{1}{3}$.

Substituting $a=\frac{1}{2},\ l=-\frac{1}{3},\ n=7$, in (I), § 359,

$$-\frac{1}{3}=\frac{1}{2}+6\,d,\ -\frac{5}{6}=6\,d,\ \text{or}\ d=-\frac{5}{36}.$$

Then the arithmetic series is $\frac{1}{2},\ \frac{13}{36},\ \frac{2}{9},\ \frac{1}{12},\ -\frac{1}{18},\ -\frac{7}{36},\ -\frac{1}{3}.$

Therefore, the required harmonic series is

$$2,\ \frac{36}{13},\ \frac{9}{2},\ 12,\ -18,\ -\frac{36}{7},\ -3.$$

381. Let x denote the harmonic mean between a and b.

Then, $\frac{1}{x}$ is the arithmetic mean between $\frac{1}{a}$ and $\frac{1}{b}$.

Then, by § 366, $\frac{1}{x}=\frac{\frac{1}{a}+\frac{1}{b}}{2}=\frac{a+b}{2\,ab}$, and $x=\frac{2\,ab}{a+b}.$

EXERCISE 164

Find the last terms of the following:

1. $\frac{5}{8},\ \frac{15}{8},\ \frac{15}{7},\ \cdots$ to 19 terms.

2. $-\frac{6}{5}, -\frac{2}{5}, -\frac{6}{25}, \cdots$ to 46 terms.

3. $-\frac{3}{4}, -\frac{4}{3}, -6, \cdots$ to 33 terms.

4. $\frac{1}{11}, \frac{4}{39}, \frac{2}{17}, \cdots$ to 11 terms.

5. $\frac{4}{7}, \frac{8}{17}, \frac{2}{5}, \cdots$ to 28 terms.

6. Insert 7 harmonic means between -4 and $\frac{12}{13}$.

7. Insert 8 harmonic means between $-\frac{2}{5}$ and $-\frac{5}{4}$.

8. Insert 6 harmonic means between $\frac{1}{2}$ and $-\frac{4}{27}$.

Find the harmonic mean between:

9. $\frac{2}{3}$ and $-\frac{3}{4}$.

10. $\dfrac{a+b}{a-b}$ and $\dfrac{a^2-b^2}{a^2+b^2}$.

11. Find the next to the last term of the harmonic progression $a, b, \cdots$ to n terms.

12. If m harmonic means are inserted between a and b, what is the third mean?

13. The sixth term of a harmonic progression is $\frac{1}{3}$, and the eleventh term $-\frac{6}{7}$. Find the fourteenth term.

14. The geometric mean between two numbers is 4, and the harmonic mean $\frac{16}{5}$. Find the numbers.

382. *If any three consecutive terms of a harmonic series be taken, the first is to the third as the first minus the second is to the second minus the third.*

Let the terms be a, b, and c; then, since $\frac{1}{a}$, $\frac{1}{b}$, and $\frac{1}{c}$ are in arithmetic progression,

$$\frac{1}{c}-\frac{1}{b}=\frac{1}{b}-\frac{1}{a}, \text{ or } \frac{b-c}{bc}=\frac{a-b}{ab}.$$

Multiplying both members by $\dfrac{ab}{b-c}$, we have

$$\frac{a}{c}=\frac{a-b}{b-c}.$$

383. Let A, G, and H denote the arithmetic, geometric, and harmonic means, respectively, between a and b.

Then, by §§ 366, 376, and 381,

$$A = \frac{a+b}{2},\ G = \sqrt{ab},\ \text{and}\ H = \frac{2ab}{a+b}.$$

But, $$\frac{a+b}{2} \times \frac{2ab}{a+b} = ab = (\sqrt{ab})^2.$$

Whence, $$A \times H = G^2,\ \text{or}\ G = \sqrt{A \times H}.$$

That is, *the geometric mean between two numbers is also the geometric mean between their arithmetic and harmonic means.*

XXVIII. THE BINOMIAL THEOREM

POSITIVE INTEGRAL EXPONENT

384. A **Series** is a succession of terms.

A **Finite Series** is one having a limited number of terms.

An **Infinite Series** is one having an unlimited number of terms.

385. In §§ 97 and 205 we gave rules for finding the square or cube of any binomial.

The **Binomial Theorem** is a formula by means of which any power of a binomial may be expanded into a series.

386. Proof of the Binomial Theorem for a Positive Integral Exponent.

The following are obtained by actual multiplication:

$$(a+x)^2 = a^2 + 2ax + x^2;$$

$$(a+x)^3 = a^3 + 3a^2x + 3ax^2 + x^3;$$

$$(a+x)^4 = a^4 + 4a^3x + 6a^2x^2 + 4ax^3 + x^4; \text{ etc.}$$

In these results, we observe the following laws:

1. The number of terms is greater by 1 than the exponent of the binomial.

2. The exponent of a in the first term is the same as the exponent of the binomial, and decreases by 1 in each succeeding term.

3. The exponent of x in the second term is 1, and increases by 1 in each succeeding term.

4. The coefficient of the first term is 1, and the coefficient of the second term is the exponent of the binomial.

5. If the coefficient of any term be multiplied by the exponent of a in that term, and the result divided by the exponent of x in the term increased by 1, the quotient will be the coefficient of the next following term.

387. If the laws of § 386 be assumed to hold for the expansion of $(a+x)^n$, where n is any positive integer, the exponent of a in the first term is n, in the second term $n-1$, in the third term $n-2$, in the fourth term $n-3$, etc.

The exponent of x in the second term is 1, in the third term 2, in the fourth term 3, etc.

The coefficient of the first term is 1; of the second term n.

Multiplying the coefficient of the second term, n, by $n-1$, the exponent of a in that term, and dividing the result by the exponent of x in the term increased by 1, or 2, we have $\frac{n(n-1)}{1 \cdot 2}$ as the coefficient of the third term; and so on.

$$\text{Then, } (a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{1 \cdot 2}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}a^{n-3}x^3 + \cdots. \tag{1}$$

Multiplying both members of (1) by $a+x$, we have

$$\begin{aligned}(a+x)^{n+1} &= a^{n+1} + na^n x + \frac{n(n-1)}{1 \cdot 2}a^{n-1}x^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}a^{n-2}x^3 + \cdots \\ &\quad + a^n x + na^{n-1}x^2 + \frac{n(n-1)}{1 \cdot 2}a^{n-2}x^3 + \cdots.\end{aligned}$$

Collecting the terms which contain like powers of a and x, we have,

$$\begin{aligned}(a+x)^{n+1} &= a^{n+1} + (n+1)a^n x + \left[\frac{n(n-1)}{1 \cdot 2} + n\right]a^{n-1}x^2 \\ &\quad + \left[\frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} + \frac{n(n-1)}{1 \cdot 2}\right]a^{n-2}x^3 + \cdots \\ &= a^{n+1} + (n+1)a^n x + n\left[\frac{n-1}{2} + 1\right]a^{n-1}x^2 \\ &\quad + \frac{n(n-1)}{1 \cdot 2}\left[\frac{n-2}{3} + 1\right]a^{n-2}x^3 + \cdots.\end{aligned}$$

$$\text{Then, } (a+x)^{n+1} = a^{n+1} + (n+1)a^n x + n\left[\frac{n+1}{2}\right]a^{n-1}x^2 + \frac{n(n-1)}{1\cdot 2}\left[\frac{n+1}{3}\right]a^{n-2}x^3 + \cdots$$

$$= a^{n+1} + (n+1)a^n x + \frac{(n+1)n}{1\cdot 2}a^{n-1}x^2 + \frac{(n+1)n(n-1)}{1\cdot 2\cdot 3}a^{n-2}x^3 + \cdots. \quad (2)$$

It will be observed that this result is in accordance with the laws of § 386; which proves that, if the laws hold for any power of $a+x$ whose exponent is a positive integer, they also hold for a power whose exponent is greater by 1.

But the laws have been shown to hold for $(a+x)^4$, and hence they also hold for $(a+x)^5$; and since they hold for $(a+x)^5$, they also hold for $(a+x)^6$; and so on.

Therefore, the laws hold when the exponent is any positive integer, and equation (1) is proved for every positive integral value of n.

Equation (1) is called the *Binomial Theorem.*

In place of the denominators $1\cdot 2$, $1\cdot 2\cdot 3$, etc., it is usual to write $\underline{|2}$, $\underline{|3}$, etc.

The symbol $\underline{|n}$, read "factorial-n," signifies the product of the natural numbers from 1 to n, inclusive.

The method of proof exemplified in § 387 is known as *Mathematical Induction.*

A more complete form of the proof of § 387, in which the fifth law of § 386 is proved for any two consecutive terms, will be found in § 447.

388. Putting $a=1$ in equation (1), § 387, we have

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{\underline{|2}}x^2 + \frac{n(n-1)(n-2)}{\underline{|3}}x^3 + \cdots.$$

389. In expanding expressions by the Binomial Theorem, it is convenient to obtain the exponents and coefficients of the terms by aid of the laws of § 386.

1. Expand $(a+x)^5$.

The exponent of a in the first term is 5, and decreases by 1 in each succeeding term.

The exponent of x in the second term is 1, and increases by 1 in each succeeding term.

The coefficient of the first term is 1; of the second, 5.

Multiplying 5, the coefficient of the second term, by 4, the exponent of a in that term, and dividing the result by the exponent of x increased by 1, or 2, we have 10 as the coefficient of the third term; and so on.

Then, $(a+x)^5 = a^5 + 5\,a^4x + 10\,a^3x^2 + 10\,a^2x^3 + 5\,ax^4 + x^5.$

It will be observed that the coefficients of terms equally distant from the ends of the expansion are equal; this law will be proved in § 391.

Thus the coefficients of the latter half of an expansion may be written out from the first half.

If the second term of the binomial is *negative*, it should be enclosed, negative sign and all, in parentheses before applying the laws; in reducing, care must be taken to apply the principles of § 96.

2. Expand $(1-x)^6$.

$$\begin{aligned}(1-x)^6 &= [1+(-x)]^6\\ &= 1^6 + 6\cdot 1^5\cdot(-x) + 15\cdot 1^4\cdot(-x)^2 + 20\cdot 1^3\cdot(-x)^3\\ &\qquad + 15\cdot 1^2\cdot(-x)^4 + 6\cdot 1\cdot(-x)^5 + (-x)^6\\ &= 1 - 6\,x + 15\,x^2 - 20\,x^3 + 15\,x^4 - 6\,x^5 + x^6.\end{aligned}$$

If the first term of the binomial is an arithmetical number, it is convenient to write the exponents at first without reduction; the result should afterwards be reduced to its simplest form.

If either term of the binomial has a coefficient or exponent other than unity, it should be enclosed in parentheses before applying the laws.

3. Expand $(3\,m^2 - \sqrt[3]{n})^4$.

$$\begin{aligned}(3\,m^2 - \sqrt[3]{n})^4 &= [(3\,m^2) + (-n^{\frac{1}{3}})]^4\\ &= (3\,m^2)^4 + 4(3\,m^2)^3(-n^{\frac{1}{3}}) + 6(3\,m^2)^2(-n^{\frac{1}{3}})^2\\ &\qquad + 4(3\,m^2)(-n^{\frac{1}{3}})^3 + (-n^{\frac{1}{3}})^4\\ &= 81\,m^8 - 108\,m^6n^{\frac{1}{3}} + 54\,m^4n^{\frac{2}{3}} - 12\,m^2n + n^{\frac{4}{3}}.\end{aligned}$$

A *trinomial* may be raised to any power by the Binomial Theorem, if two of its terms be enclosed in parentheses, and regarded as a single term; but for second powers, the method of § 204 is shorter.

4. Expand $(x^2-2x-2)^4$.

$$
\begin{aligned}
(x^2-2x-2)^4 &= [(x^2-2x)+(-2)]^4 \\
&= (x^2-2x)^4+4(x^2-2x)^3(-2)+6(x^2-2x)^2(-2)^2 \\
&\qquad +4(x^2-2x)(-2)^3+(-2)^4 \\
&= x^8-8x^7+24x^6-32x^5+16x^4 \\
&\qquad -8(x^6-6x^5+12x^4-8x^3) \\
&\qquad +24(x^4-4x^3+4x^2)-32(x^2-2x)+16 \\
&= x^8-8x^7+16x^6+16x^5-56x^4-32x^3+64x^2+64x+16.
\end{aligned}
$$

EXERCISE 165

Expand the following:

1. $(a+x)^4$.
2. $(n+1)^6$.
3. $(1-y)^5$.
4. $(a-x)^7$.
5. $(x^2y+z^3)^5$.
6. $(x+2y)^4$.
7. $(2-a^2)^5$.
8. $(3a^4-b^3)^4$.
9. $(a^{4m}+x^{3n})^5$.
10. $(2x^2-y^3)^6$.
11. $(a+n)^8$.
12. $(a^{-2}+\sqrt[4]{a^5})^6$.
13. $(x^{\frac{2}{3}}+3)^5$.
14. $(1-x^2)^8$.
15. $\left(\sqrt{a}-\frac{1}{\sqrt[6]{b^5}}\right)^6$.
16. $(m^{\frac{5}{2}}+4n^{-\frac{3}{4}})^4$.
17. $\left(x^{\frac{2}{3}}y^{-\frac{1}{3}}+\frac{1}{z^3}\right)^7$.
18. $\left(m^{-\frac{2}{5}}+\frac{n^{-4}}{2}\right)^5$.
19. $(5x^{-5}-\sqrt[3]{n})^4$.
20. $(2x^{\frac{1}{3}}+y^{-\frac{3}{2}})^7$.
21. $\left(\frac{x^5}{3}-y^4z\right)^5$.
22. $\left(3x^{\frac{4}{3}}-\frac{1}{2x^{\frac{3}{4}}}\right)^4$.
23. $(3a^{-\frac{1}{4}}+\sqrt[6]{a})^6$.
24. $\left(3\sqrt{a^5}-\frac{1}{3\sqrt[3]{x^5}}\right)^4$.
25. $\left(\sqrt[5]{x}-\frac{\sqrt{n^3}}{2}\right)^6$.
26. $\left(\frac{2m^2}{n}+\frac{n^2}{3m}\right)^4$.
27. $\left(\sqrt[4]{a}-\frac{1}{3\sqrt[5]{b^3}}\right)^5$.
28. $(a-b)^9$.
29. $(a+1)^{10}$.
30. $(x^2+x+1)^4$.
31. $(2-x+x^2)^4$.
32. $(x^2-3x-1)^4$.
33. $(3x^2+x-2)^4$.
34. $(1-x+x^2)^5$.
35. $(x^2+x-3)^5$.

390. *To find the rth or general term in the expansion of* $(a+x)^n$.

The following laws hold for any term in the expansion of $(a+x)^n$, in equation (1), § 387:

1. The exponent of x is less by 1 than the number of the term.
2. The exponent of a is n minus the exponent of x.
3. The last factor of the numerator is greater by 1 than the exponent of a.
4. The last factor of the denominator is the same as the exponent of x.

Therefore in the rth term, the exponent of x will be $r-1$.

The exponent of a will be $n-(r-1)$, or $n-r+1$.

The last factor of the numerator will be $n-r+2$.

The last factor of the denominator will be $r-1$.

Hence, the rth term

$$=\frac{n(n-1)(n-2)\cdots(n-r+2)}{1\cdot 2\cdot 3\cdots(r-1)}a^{n-r+1}x^{r-1}. \qquad (1)$$

In finding any term of an expansion, it is convenient to obtain the coefficient and exponents of the terms by the above laws.

Ex. Find the 8th term of $(3a^{\frac{1}{2}}-b^{-1})^{11}$.

We have, $(3a^{\frac{1}{2}}-b^{-1})^{11}=[(3a^{\frac{1}{2}})+(-b^{-1})]^{11}$.
In this case, $n=11$, $r=8$.
The exponent of $(-b^{-1})$ is $8-1$, or 7.
The exponent of $(3a^{\frac{1}{2}})$ is $11-7$, or 4.
The first factor of the numerator is 11, and the last factor $4+1$, or 5.
The last factor of the denominator is 7.

$$\text{Then, the 8th term} = \frac{11\cdot 10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}(3a^{\frac{1}{2}})^4(-b^{-1})^7$$

$$=330(81a^2)(-b^{-7})=-26730a^2b^{-7}.$$

If the second term of the binomial is negative, it should be enclosed, sign and all, in parentheses before applying the laws.

If either term of the binomial has a coefficient or exponent other than unity, it should be enclosed in parentheses before applying the laws.

EXERCISE 166

Find the:

1. 4th term of $(a+x)^7$.
2. 6th term of $(n+1)^{11}$.
3. 5th term of $(a-b)^8$.
4. 7th term of $(1-a^2)^9$.
5. 8th term of $(x^4+y^3)^{12}$.
6. 5th term of $(a^{\frac{1}{3}}+2x^{-\frac{1}{2}})^{16}$.
7. 9th term of $(x^m-x^n)^{13}$.
8. 10th term of $\left(\dfrac{a^3}{b}+\dfrac{b^3}{a}\right)^{18}$.
9. 10th term of $\left(\sqrt[5]{m^2}-\dfrac{n^{-\frac{2}{3}}}{2}\right)^{14}$.
10. 6th term of $(x^4-4y^{\frac{3}{4}})^{15}$.
11. 7th term of $\left(m^{-\frac{2}{3}}+\dfrac{3}{\sqrt[3]{n}}\right)^{10}$.
12. 4th term of $(m^{-4}-5mn)^{16}$.
13. 9th term of $\left(a^{\frac{1}{6}}+\dfrac{1}{4\sqrt[4]{x^3}}\right)^{15}$.
14. 8th term of $\left(\sqrt{a^3}-\dfrac{b^{-3}}{3}\right)^{14}$.
15. Middle term of $\left(3a^5+\dfrac{b^{\frac{3}{4}}}{2}\right)^{12}$.

391. Multiplying both terms of the coefficient, in (1), § 390, by the product of the natural numbers from 1 to $n-r+1$, inclusive, the coefficient of the rth term becomes

$$\frac{n(n-1)\cdots(n-r+2)\cdot(n-r+1)\cdots 2\cdot 1}{\underline{|r-1}\times 1\cdot 2\cdots(n-r+1)}=\frac{\underline{|n}}{\underline{|r-1}\;\underline{|n-r+1}}.$$

Since the number of terms in the expansion is $n+1$, the rth term from the end is the $(n-r+2)$th from the beginning.

Then, to find the coefficient of the rth term from the end, we put in the above formula $n-r+2$ for r.

Then, the coefficient of the rth term from the end is

$$\frac{\underline{|n}}{\underline{|n-r+2-1}\;\underline{|n-(n-r+2)+1}},\text{ or }\frac{\underline{|n}}{\underline{|n-r+1}\;\underline{|r-1}}.$$

Hence, *in the expansion of* $(a+x)^n$, *the coefficients of terms equidistant from the ends of the expansion are equal.*

XXIX. UNDETERMINED COEFFICIENTS

392. Infinite Series (§ 384) may be developed by Division, or by Evolution.

Let it be required, for example, to divide 1 by $1-x$.

$$\begin{array}{l}
1-x)1(1+x+x^2+\cdots \\
\quad\;\; \underline{1-x} \\
\qquad\quad x \\
\qquad\quad \underline{x-x^2} \\
\qquad\qquad\;\; x^2
\end{array}$$

Then, $\dfrac{1}{1-x}=1+x+x^2+x^3+\cdots.$ (1)

Again, let it be required to find the square root of $1+x$.

$$\begin{array}{l|l}
1+x & 1+\dfrac{x}{2}-\dfrac{x^2}{8}+\cdots \\
\underline{1} & \\
2+\dfrac{x}{2} & x \\
 & \underline{x+\dfrac{x^2}{4}} \\
2+x-\dfrac{x^2}{8} & -\dfrac{x^2}{4}
\end{array}$$

Then, $\sqrt{1+x}=1+\dfrac{x}{2}-\dfrac{x^2}{8}+\cdots.$ (2)

It should be observed that the series, in (1) and (2), do not give the values of the first members for every value of x; thus, if x is a very large number, they evidently do not do so.

EXERCISE 167

Expand each of the following to four terms:

1. $\dfrac{3+4x}{1+2x}$.

2. $\dfrac{1-5x^2}{1-5x-2x^2}$.

3. $\dfrac{4x}{2+6x-x^2}$.

4. $\dfrac{2+4x-5x^2}{3-6x^2+7x^3}$.

5. $\sqrt{1+6x}$.

6. $\sqrt{1-2x}$.

7. $\sqrt{1+a}$.
9. $\sqrt{x^2+xy+y^2}$.
11. $\sqrt[3]{x^3+1}$.
8. $\sqrt{1-5a}$.
10. $\sqrt{9a^2+b}$.
12. $\sqrt[3]{a^6-3b^3}$.

CONVERGENCY AND DIVERGENCY OF SERIES

393. An infinite series is said to be *Convergent* when the sum of the first n terms approaches a fixed finite number as a limit (§ 318), when n is indefinitely increased.

An infinite series is said to be *Divergent* when the sum of the first n terms can be made numerically greater than any assigned number, however great, by taking n sufficiently great.

394. Consider, for example, the infinite series

$$1+x+x^2+x^3+\cdots.$$

I. Suppose $x=x_1$, where x_1 is numerically <1.

The sum of the first n terms is now

$$1+x_1+x_1^2+\cdots+x_1^{n-1}=\frac{1-x_1^n}{1-x_1} \text{ (§ 103).}$$

If n be indefinitely increased, x_1^n decreases indefinitely in absolute value, and approaches the limit 0.

Then the fraction $\dfrac{1-x_1^n}{1-x_1}$ approaches the limit $\dfrac{1}{1-x_1}$.

That is, the sum of the first n terms approaches a fixed finite number as a limit, when n is indefinitely increased.

Hence, the series is *convergent* when x is numerically <1.

II. Suppose $x=1$.

In this case, each term of the series is equal to 1, and the sum of the first n terms is equal to n; and this sum can be made to exceed any assigned number, however great, by taking n sufficiently great.

Hence, the series is *divergent* when $x=1$.

III. Suppose $x=-1$.

In this case, the series takes the form $1-1+1-1+\cdots$, and the sum of the first n terms is either 1 or 0 according as n is odd or even.

Hence, the series is neither convergent nor divergent when $x = -1$.

An infinite series which is neither convergent nor divergent is called an *Oscillating Series.*

IV. Suppose $x = x_1$, where x_1 is numerically > 1.

The sum of the first n terms is now

$$1 + x_1 + x_1^2 + \cdots + x_1^{n-1} = \frac{x_1^n - 1}{x_1 - 1} \text{ (§ 103).}$$

By taking n sufficiently great, $\frac{x_1^n - 1}{x_1 - 1}$ can be made to numerically exceed any assigned number, however great.

Hence, the series is *divergent* when x is numerically > 1.

395. Consider the infinite series

$$1 + x + x^2 + x^3 + \cdots,$$

developed by the fraction $\frac{1}{1-x}$ (§ 392).

Let $x = .1$, in which case the series is convergent (§ 394).

The series now takes the form $1 + .1 + .01 + .001 + \cdots$, while the value of the fraction is $\frac{1}{.9}$, or $\frac{10}{9}$.

In this case, however great the number of terms taken, their sum will never exactly equal $\frac{10}{9}$.

But the sum approaches this value as a limit; for the series is a decreasing geometric progression, whose first term is 1, and ratio .1; and, by § 373, its sum to infinity is $\frac{1}{1-.1}$, or $\frac{10}{9}$.

Thus, if an infinite series is *convergent,* the greater the number of terms taken, the more nearly does their sum approach to the value of the expression from which the series was developed.

Again, let $x = 10$, in which case the series is divergent.

The series now takes the form $1 + 10 + 100 + 1000 + \cdots$, while the value of the fraction is $\frac{1}{1-10}$, or $-\frac{1}{9}$.

In this case the greater the number of terms taken, the more does their sum diverge from the value $-\frac{1}{3}$.

Thus, if an infinite series is *divergent*, the greater the number of terms taken, the more does their sum diverge from the value of the expression from which the series was developed.

It follows from the above that *an infinite series cannot be used for the purposes of demonstration, if it is divergent.*

THE THEOREM OF UNDETERMINED COEFFICIENTS

396. An important method for expanding expressions into series is based on the following theorem:

If the series $A+Bx+Cx^2+Dx^3+\cdots$ is always equal to the series $A'+B'x+C'x^2+D'x^3+\cdots$, when x has any value which makes both series convergent, the coefficients of like powers of x in the series will be equal; that is, $A=A'$, $B=B'$, $C=C'$, etc.

For the equation

$$A+Bx+Cx^2+Dx^3+\cdots=A'+B'x+C'x^2+D'x^3+\cdots \tag{1}$$

is satisfied when x has any value which makes both members convergent.

But both members are convergent when $x=0$; for the sum of all the terms of the infinite series $a+bx+cx^2+dx^3+\cdots$ is equal to a when $x=0$.

Then, the equation (1) is satisfied when $x=0$.

Putting $x=0$, we have $A=A'$.

Subtracting A from the first member of the equation, and its equal A' from the second member, we obtain

$$Bx+Cx^2+Dx^3+\cdots=B'x+C'x^2+D'x^3+\cdots. \tag{2}$$

Dividing each term by x,

$$B+Cx+Dx^2+\cdots=B'+C'x+D'x^2+\cdots. \tag{3}$$

This equation also is satisfied when x has any value which makes both members convergent; and putting $x=0$, we have

$$B=B'.$$

In like manner, we may prove $C = C'$, $D = D'$, etc.

The proof of § 396 is open to objection in one respect.

We know that (2) has the same roots as (1), including the root 0; but when we divide by x, all that we know about the resulting equation is that it has the same roots as (2), *except the root* 0.

Thus, we do not know that 0 is a root of (3), though we assume it in proving that $B = B'$.

A more rigorous proof of the Theorem of Undetermined Coefficients will be found in § 450.

397. The theorem of § 396 holds when either or both of the given series are finite.

EXPANSION OF FRACTIONS

398. 1. Expand $\dfrac{2-3x^2-x^3}{1-2x+3x^2}$ in ascending powers of x.

Assume
$$\frac{2-3x^2-x^3}{1-2x+3x^2} = A + Bx + Cx^2 + Dx^3 + Ex^4 + \cdots, \tag{1}$$

where A, B, C, D, E, $\cdots$, are numbers independent of x.

Clearing of fractions, and collecting the terms in the second member involving like powers of x, we have

$$2 - 3x^2 - x^3 = A + \begin{array}{r|l} B & \\ -2A & \\ & \end{array} x + \begin{array}{r|l} C & \\ -2B & \\ +3A & \end{array} x^2 + \begin{array}{r|l} D & \\ -2C & \\ +3B & \end{array} x^3 + \begin{array}{r|l} E & \\ -2D & \\ +3C & \end{array} x^4 + \cdots. \tag{2}$$

A vertical line, called a *bar*, is often used in place of parentheses.

Thus, $\begin{array}{r|l} +\ B & \\ -2A & \end{array} x$ is equivalent to $(B-2A)x$.

The second member of (1) must express the value of the fraction for every value of x which makes the series convergent (§ 395); and therefore equation (2) is satisfied when x has any value which makes the second member convergent.

Then, by § 397, the coefficients of like powers of x in (2) must be equal; that is,

$$\begin{aligned}
A &= 2.\\
B - 2A &= 0;\ \text{or, } B = 2A = 4.\\
C - 2B + 3A &= -3;\ \text{or, } C = 2B - 3A - 3 = -1.\\
D - 2C + 3B &= -1;\ \text{or, } D = 2C - 3B - 1 = -15.\\
E - 2D + 3C &= 0;\ \text{or, } E = 2D - 3C = -27;\ \text{etc.}
\end{aligned}$$

Substituting these values in (1), we have

$$\frac{2-3x^2-x^3}{1-2x+3x^2}=2+4x-x^2-15x^3-27x^4-\cdots.$$

The result may be verified by division.

The series expresses the value of the fraction only for such values of x as make it convergent (§ 395).

If the numerator and denominator contain only *even* powers of x, the operation may be abridged by assuming a series containing only the even powers of x.

Thus, if the fraction were $\frac{2+4x^2-x^4}{1-3x^2+5x^4}$, we should assume it equal to $A+Bx^2+Cx^4+Dx^6+Ex^8+\cdots$.

In like manner, if the numerator contains only *odd* powers of x, and the denominator only even powers, we should assume a series containing only the odd powers of x.

If every term of the numerator contains x, we may assume a series commencing with the lowest power of x in the numerator.

If every term of the denominator contains x, we determine by actual division what power of x will occur in the first term of the expansion, and then assume the fraction equal to a series commencing with this power of x, the exponents of x in the succeeding terms increasing by unity as before.

2. Expand $\frac{1}{3x^2-x^3}$ in ascending powers of x.

Dividing 1 by $3x^2$, the quotient is $\frac{x^{-2}}{3}$; we then assume,

$$\frac{1}{3x^2-x^3}=Ax^{-2}+Bx^{-1}+C+Dx+Ex^2+\cdots. \qquad (3)$$

Clearing of fractions,

$$1=3A\left.\begin{matrix}+3B\\-A\end{matrix}\right|x\left.\begin{matrix}+3C\\-B\end{matrix}\right|x^2\left.\begin{matrix}+3D\\-C\end{matrix}\right|x^3\left.\begin{matrix}+3E\\-D\end{matrix}\right|x^4+\cdots.$$

Equating coefficients of like powers of x,

$3A=1,\ 3B-A=0,\ 3C-B=0,\ 3D-C=0,\ 3E-D=0$; etc.

Whence, $A=\frac{1}{3},\ B=\frac{1}{9},\ C=\frac{1}{27},\ D=\frac{1}{81},\ E=\frac{1}{243},$ etc.

Substituting in (3), $\dfrac{1}{3x^2-x^3}=\dfrac{x^{-2}}{3}+\dfrac{x^{-1}}{9}+\dfrac{1}{27}+\dfrac{x}{81}+\dfrac{x^2}{243}+\cdots.$

In Ex. 1, $E=2D-3C$; that is, the coefficient of x^4 equals twice the coefficient of the preceding term, minus three times the coefficient of the next but one preceding.

It is evident that this law holds for the succeeding terms; thus, the coefficient of x^5 is $2\times(-27)-3\times(-15)$, or -9.

After the law of coefficients has been found in any expansion, the terms may be found more easily than by long division; and for this reason the method of § 398 is to be preferred when a large number of terms is required.

The law for Ex. 2 is that each coefficient is one-third the preceding.

EXERCISE 168

Expand each of the following to five terms in ascending powers of x:

1. $\dfrac{3+2x}{1-x}$.
2. $\dfrac{1-6x}{1+4x}$.
3. $\dfrac{4+x^2}{1-3x^2}$.
4. $\dfrac{2x}{3-5x^2}$.
5. $\dfrac{1+4x-x^2}{1-x+3x^2}$.
6. $\dfrac{2-x+3x^2}{1+2x^2}$.
7. $\dfrac{1-4x^2}{1+4x-x^2}$.
8. $\dfrac{x-5x^3-7x^4}{1-2x-4x^2}$.
9. $\dfrac{x^2-3x^3}{3-x-2x^3}$.
10. $\dfrac{4+x-5x^2}{2-x+3x^2}$.
11. $\dfrac{3+4x^3}{2+3x^2-x^3}$.
12. $\dfrac{3}{2x^3+x^4}$.
13. $\dfrac{2+4x-3x^2}{x^2-5x^3+x^4}$.
14. $\dfrac{1-6x^2+4x^3}{x+x^2-2x^3}$.
15. $\dfrac{1-2x+3x^3}{2x^3+4x^5+x^6}$.

EXPANSION OF SURDS

399. *Ex.* Expand $\sqrt{1-x}$ in ascending powers of x.

Assume $\sqrt{1-x}=A+Bx+Cx^2+Dx^3+Ex^4+\cdots.$ (1)

Squaring both members, we have by § 204,

$$1-x=A^2\left.\begin{array}{l}\\+2AB\\\end{array}\right|x+\left.\begin{array}{r}B^2\\+2AC\\\end{array}\right|x^2\left.\begin{array}{r}\\+2AD\\+2BC\end{array}\right|x^3+\left.\begin{array}{r}C^2\\+2AE\\+2BD\end{array}\right|x^4+\cdots.$$

Equating coefficients of like powers of x,

$$A^2 = 1;\ \text{or},\ A = 1.$$

$$2AB = -1;\ \text{or},\ B = -\frac{1}{2A} = -\frac{1}{2}.$$

$$B^2 + 2AC = 0;\ \text{or},\ C = -\frac{B^2}{2A} = -\frac{1}{8}.$$

$$2AD + 2BC = 0;\ \text{or},\ D = -\frac{BC}{A} = -\frac{1}{16}.$$

$$C^2 + 2AE + 2BD = 0;\ \text{or},\ E = -\frac{C^2 + 2BD}{2A} = -\frac{5}{128};\ \text{etc.}$$

Substituting these values in (1), we have

$$\sqrt{1-x} = 1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{16} - \frac{5x^4}{128} - \cdots.$$

The result may be verified by Evolution.

The series expresses the value of $\sqrt{1-x}$ only for such values of x as make it convergent.

EXERCISE 169

Expand each of the following to five terms in ascending powers of x:

1. $\sqrt{1+2x}$.
2. $\sqrt{1-3x}$.
3. $\sqrt{1-4x+x^2}$.
4. $\sqrt{1+x-x^2}$.
5. $\sqrt[3]{1+6x}$.
6. $\sqrt[3]{1-x-2x^2}$.

PARTIAL FRACTIONS

400. If the denominator of a fraction can be resolved into factors, each of the first degree in x, and the numerator is of a lower degree than the denominator, the Theorem of Undetermined Coefficients enables us to express the given fraction as the sum of two or more *partial fractions*, whose denominators are factors of the given denominator, and whose numerators are independent of x.

401. Case I. *No factors of the denominator equal.*

1. Separate $\dfrac{19x+1}{(3x-1)(5x+2)}$ into partial fractions.

Assume $$\frac{19x+1}{(3x-1)(5x+2)}=\frac{A}{3x-1}+\frac{B}{5x+2},\tag{1}$$

where A and B are numbers independent of x.

Clearing of fractions, $19x+1=A(5x+2)+B(3x-1)$.

Or, $$19x+1=(5A+3B)x+2A-B.\tag{2}$$

The second member of (1) must express the value of the given fraction for every value of x.

Hence, equation (2) is satisfied by every value of x; and by § 397, the coefficients of like powers of x in the two members are equal.

That is, $$5A+3B=19,$$

and $$2A-\quad B=1.$$

Solving these equations, we obtain $A=2$ and $B=3$.

Substituting in (1), $\dfrac{19x+1}{(3x-1)(5x+2)}=\dfrac{2}{3x-1}+\dfrac{3}{5x+2}.$

The result may be verified by finding the sum of the partial fractions.

2. Separate $\dfrac{x+4}{2x-x^2-x^3}$ into partial fractions.

The factors of $2x-x^2-x^3$ are x, $1-x$, and $2+x$ (§ 116).

Assume then $$\frac{x+4}{2x-x^2-x^3}=\frac{A}{x}+\frac{B}{1-x}+\frac{C}{2+x}.$$

Clearing of fractions, we have

$$x+4=A(1-x)(2+x)+Bx(2+x)+Cx(1-x).$$

This equation, being satisfied by every value of x, is satisfied when $x=0$.
Putting $x=0$, we have $4=2A$, or $A=2$.
Again, the equation is satisfied when $x=1$.
Putting $x=1$, we have $5=3B$, or $B=\frac{5}{3}$.
The equation is also satisfied when $x=-2$.
Putting $x=-2$, we have $2=-6C$, or $C=-\frac{1}{3}$.

Then, $$\frac{x+4}{2x-x^2-x^3}=\frac{2}{x}+\frac{\frac{5}{3}}{1-x}+\frac{-\frac{1}{3}}{2+x}=\frac{2}{x}+\frac{5}{3(1-x)}-\frac{1}{3(2+x)}.$$

To find the value of A, in Ex. 2, we give to x *such a value as will make the coefficients of B and C equal to zero;* and we proceed in a similar manner to find the values of B and C.

This method of finding A, B, and C is usually shorter than that used in Ex. 1.

EXERCISE 170

Separate the following into partial fractions:

1. $\dfrac{27x-6}{9x^2-4}$.
3. $\dfrac{x^2-48}{x^3-16x}$.
5. $\dfrac{5ax^2-2a^2x-8a^3}{x^3+3ax^2-4a^2x}$.
2. $\dfrac{23x+25}{6x^2+5x}$.
4. $\dfrac{6x-11}{6x^2+13x+6}$.
6. $\dfrac{10-9x}{5x^2-14x+8}$.
7. $\dfrac{12+17x-2x^2}{(1+3x)(9+6x-8x^2)}$.
8. $\dfrac{4+14x-2x^2}{(x^2-5x)(x^2-4)}$.

402. Case II. *All the factors of the denominator equal.*

Let it be required to separate $\dfrac{x^2-11x+26}{(x-3)^3}$ into partial fractions.

Substituting $y+3$ for x, the fraction becomes

$$\frac{(y+3)^2-11(y+3)+26}{y^3}=\frac{y^2-5y+2}{y^3}=\frac{1}{y}-\frac{5}{y^2}+\frac{2}{y^3}.$$

Replacing y by $x-3$, the result takes the form

$$\frac{1}{x-3}-\frac{5}{(x-3)^2}+\frac{2}{(x-3)^3}.$$

This shows that the given fraction can be expressed as the sum of three partial fractions, whose numerators are independent of x, and whose denominators are the powers of $x-3$ beginning with the first and ending with the third.

Similar considerations hold with respect to any example under Case II; the number of partial fractions in any case being the same as the number of equal factors in the denominator of the given fraction.

Ex. Separate $\dfrac{6x+5}{(3x+5)^2}$ into partial fractions.

In accordance with the above principle, we assume the given fraction equal to the sum of *two* partial fractions, whose denominators are the powers of $3x+5$ beginning with the first and ending with the *second*.

That is, $$\frac{6x+5}{(3x+5)^2}=\frac{A}{3x+5}+\frac{B}{(3x+5)^2}.$$

Clearing of fractions, $6x+5=A(3x+5)+B$.

$$=3Ax+5A+B.$$

Equating coefficients of like powers of x,

$$3A=6,$$

and $$5A+B=5.$$

Solving these equations, $A=2$ and $B=-5$.

Whence, $$\frac{6x+5}{(3x+5)^2}=\frac{2}{3x+5}-\frac{5}{(3x+5)^2}.$$

EXERCISE 171

Separate the following into partial fractions:

1. $\dfrac{24x+2}{9x^2+12x+4}$.
2. $\dfrac{2x^2-11x+3}{(x-4)^3}$.
3. $\dfrac{12x^2+7x-1}{(1+3x)^3}$.
4. $\dfrac{6x^2+12x-10}{(3+2x)^3}$.
5. $\dfrac{16x^2-19}{(4x-3)^3}$.
6. $\dfrac{x^3-2x^2-7x}{(x+1)^4}$.
7. $\dfrac{2x^3-13x^2+24x-15}{(x-2)^4}$.
8. $\dfrac{18x+54x^2+27x^3}{(2+3x)^4}$.

403. Case III. *Some of the factors of the denominator equal.*

Ex. Separate $\dfrac{x^2-4x+3}{x(x+1)^2}$ into partial fractions.

The method in Case III is a combination of the methods of Cases I and II; we assume,
$$\frac{x^2-4x+3}{x(x+1)^2}=\frac{A}{x}+\frac{B}{x+1}+\frac{C}{(x+1)^2}.$$

Clearing of fractions,

$$x^2-4x+3=A(x+1)^2+Bx(x+1)+Cx$$
$$=(A+B)x^2+(2A+B+C)x+A.$$

Equating coefficients of like powers of x,

$$A + B = 1,$$

$$2A + B + C = -4,$$

and $$A = 3.$$

Solving these equations, $A = 3$, $B = -2$, and $C = -8$.

Whence, $$\frac{x^2 - 4x + 3}{x(x+1)^2} = \frac{3}{x} - \frac{2}{x+1} - \frac{8}{(x+1)^2}.$$

The following general rule for Case III will be found convenient:

A fraction of the form $\dfrac{X}{(x+a)(x+b)\cdots(x+m)^r\cdots}$ should be assumed equal to

$$\frac{A}{x+a} + \frac{B}{x+b} + \cdots + \frac{E}{x+m} + \frac{F}{(x+m)^2} + \cdots + \frac{K}{(x+m)^r} + \cdots.$$

Single factors like $x + a$ and $x + b$ having single partial fractions corresponding, arranged as in Case I; and repeated factors like $(x + m)^r$ having r partial fractions corresponding, arranged as in Case II.

EXERCISE 172

Separate the following into partial fractions:

1. $\dfrac{3x^2 - x + 27}{x(x+3)^2}$.

2. $\dfrac{3x^3 + 7x^2 + 24x - 16}{x^3(x-4)}$.

3. $\dfrac{14x^2 - 53x - 4}{(3x+2)(2x-3)^2}$.

4. $\dfrac{4x^3 - x^2 - 7x - 4}{x^2(x+1)^2}$.

5. $\dfrac{-4x^3 + 29x^2 - 36x - 9}{x(x-1)(x-3)^2}$.

6. $\dfrac{7 - 13x - 4x^2}{(8x^2 - 2x - 3)(2x+1)}$.

404. If the degree of the numerator is equal to, or greater than, that of the denominator, the preceding methods are inapplicable.

In such a case, we divide the numerator by the denominator until a remainder is obtained which is of a lower degree than the denominator.

Ex. Separate $\dfrac{x^3 - 3x^2 - 1}{x^2 - x}$ into an integral expression and partial fractions.

Dividing $x^3 - 3x^2 - 1$ by $x^2 - x$, the quotient is $x - 2$, and the remainder $-2x - 1$; we then have

$$\frac{x^3 - 3x^2 - 1}{x^2 - x} = x - 2 + \frac{-2x - 1}{x^2 - x}. \qquad (1)$$

We can now separate $\frac{-2x-1}{x^2-x}$ into partial fractions by the method of Case I; the result is $\frac{1}{x} - \frac{3}{x-1}$.

Substituting in (1), $\frac{x^3 - 3x^2 - 1}{x^2 - x} = x - 2 + \frac{1}{x} - \frac{3}{x-1}$.

Another way to solve the above example is to combine the methods of §§ 398 and 401, and assume the given fraction equal to

$$Ax + B + \frac{C}{x} + \frac{D}{x-1}.$$

EXERCISE 173

Separate each of the following into an integral expression and two or more partial fractions:

1. $\frac{12x^3 - 17x^2 + 7}{(x-2)(3x+1)}$.

2. $\frac{2x^3 + 14x^2 + 30x + 25}{(x+3)^3}$.

3. $\frac{x^5 - 4x^4 + 2x^2 + 7x - 4}{x^3(x-1)}$.

4. $\frac{x^5 - 2x^4 - 5x^3 - 5x - 3}{x^2(x+1)^2}$.

5. $\frac{2x^6 - 8x^5 + 2x^4 - 5x^3 + 12x^2 - x + 4}{x^3(x-4)}$.

405. If the denominator of a fraction can be resolved into factors partly of the first and partly of the second, or all of the second degree, in x, and the numerator is of a lower degree than the denominator, the Theorem of Undetermined Coefficients enables us to express the given fraction as the sum of two or more partial fractions, whose denominators are factors of the given denominator, and whose numerators are independent of x in the case of fractions corresponding to factors of the first degree, and of the form $Ax + B$ in the case of fractions corresponding to factors of the second degree.

The only exceptions occur when the factors of the denominator are of the second degree and all equal.

Ex. Separate $\dfrac{1}{x^3+1}$ into partial fractions.

The factors of the denominator are $x+1$ and x^2-x+1.

Assume then $$\frac{1}{x^3+1}=\frac{A}{x+1}+\frac{Bx+C}{x^2-x+1}. \quad (1)$$

Clearing of fractions, $1=A(x^2-x+1)+(Bx+C)(x+1)$.

Or, $$1=(A+B)x^2+(-A+B+C)x+A+C.$$

Equating coefficients of like powers of x,

$$A+B=0,$$
$$-A+B+C=0,$$
and $$A+C=1.$$

Solving these equations, $A=\frac{1}{3}$, $B=-\frac{1}{3}$, and $C=\frac{2}{3}$.

Substituting in (1), $\dfrac{1}{x^3+1}=\dfrac{1}{3(x+1)}-\dfrac{x-2}{3(x^2-x+1)}$.

EXERCISE 174

Separate the following into partial fractions:

1. $\dfrac{3x^2-4x+4}{x^3-1}$.
2. $\dfrac{10+3x-11x^2}{(3x-2)(x^2-x+2)}$.
3. $\dfrac{x^2+3x-5}{(4x+5)(x^2-3)}$.
4. $\dfrac{18x^2+6x+8}{27x^3+8}$.
5. $\dfrac{4x^3-5x^2+6x+3}{x^4-1}$.
6. $\dfrac{3-8x-4x^3}{x^4+5x^2+6}$.

REVERSION OF SERIES

406. To *revert* a given series $y=a+bx^m+cx^n+\cdots$ is to express x as a series proceeding in ascending powers of y.

Ex. Revert the series $y=2x-3x^2+4x^3-5x^4+\cdots$.

Assume $$x=Ay+By^2+Cy^3+Dy^4+\cdots. \quad (1)$$

Substituting in this the given value of y,

$$\begin{aligned} x = A(&2x - 3x^2 + 4x^3 - 5x^4 + \cdots) \\ &+ B(4x^2 + 9x^4 - 12x^3 + 16x^4 + \cdots) \\ &+ C(8x^3 - 36x^4 + \cdots) + D(16x^4 + \cdots) + \cdots. \end{aligned}$$

That is, $$x = \begin{array}{r|r|r|l} 2Ax - 3A & x^2 + 4A & x^3 - 5A & x^4 + \cdots. \\ +4B & -12B & +25B & \\ & +8C & -36C & \\ & & +16D & \end{array}$$

Equating coefficients of like powers of x,

$$2A = 1;$$
$$-3A + 4B = 0;$$
$$4A - 12B + 8C = 0;$$
$$-5A + 25B - 36C + 16D = 0; \text{ etc.}$$

Solving, $$A = \frac{1}{2},\ B = \frac{3}{8},\ C = \frac{5}{16},\ D = \frac{35}{128};\ \text{etc.}$$

Substituting in (1), $x = \frac{1}{2}y + \frac{3}{8}y^2 + \frac{5}{16}y^3 + \frac{35}{128}y^4 + \cdots.$

If the even powers of x are wanting in the given series, the operation may be abridged by assuming x equal to a series containing only the *odd* powers of y.

EXERCISE 175

Revert each of the following to four terms:

1. $y = x + 3x^2 + 5x^3 + 7x^4 + \cdots.$
2. $y = x - 2x^2 + 3x^3 - 4x^4 + \cdots.$
3. $y = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots.$
4. $y = 2x + 5x^2 + 8x^3 + 11x^4 + \cdots.$
5. $y = \frac{x}{2} - \frac{x^2}{4} + \frac{x^3}{6} - \frac{x^4}{8} + \cdots.$
6. $y = \frac{x}{\lfloor 2} + \frac{x^2}{\lfloor 3} + \frac{x^3}{\lfloor 4} + \frac{x^4}{\lfloor 5} + \cdots.$
7. $y = 2x - 4x^3 + 6x^5 - 8x^7 + \cdots.$
8. $y = \frac{x}{2} + \frac{x^3}{4} + \frac{x^5}{6} + \frac{x^7}{8} + \cdots.$

XXX. THE BINOMIAL THEOREM

FRACTIONAL AND NEGATIVE EXPONENTS

407. It was proved in § 387 that, if n is a positive integer,

$$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{1\cdot 2}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}x^3 + \cdots.$$

If n is a negative integer, or a positive or negative fraction, the series in the second member is infinite; for no one of the expressions $n-1$, $n-2$, etc., can equal zero; in this case, the series gives the value of $(a+x)^n$, provided it is convergent.

As a rigorous proof of the Binomial Theorem for Fractional and Negative Exponents is too difficult for pupils in preparatory schools, the author has thought best to omit it; any one desiring a rigorous algebraic proof of the theorem, will find it in the author's Advanced Course in Algebra, § 575.

408. Examples.

In expanding expressions by the Binomial Theorem when the exponent is fractional or negative, the exponents and coefficients of the terms may be found by the laws of § 386, which hold for all values of the exponent.

1. Expand $(a+x)^{\frac{2}{3}}$ to five terms.

The exponent of a in the first term is $\frac{2}{3}$, and decreases by 1 in each succeeding term.

The exponent of x in the second term is 1, and increases by 1 in each succeeding term.

The coefficient of the first term is 1; of the second term, $\frac{2}{3}$.

Multiplying $\frac{2}{3}$, the coefficient of the second term, by $-\frac{1}{3}$, the exponent of a in that term, and dividing the product by the exponent of x increased by 1, or 2, we have $-\frac{1}{9}$ as the coefficient of the third term; and so on.

Then, $\quad (a+x)^{\frac{2}{3}} = a^{\frac{2}{3}} + \frac{2}{3}a^{-\frac{1}{3}}x - \frac{1}{9}a^{-\frac{4}{3}}x^2 + \frac{4}{81}a^{-\frac{7}{3}}x^3 - \frac{7}{243}a^{-\frac{10}{3}}x^4 + \cdots.$

2. Expand $(1+2x^{-\frac{1}{2}})^{-2}$ to five terms.

Enclosing $2x^{-\frac{1}{2}}$ in parentheses, we have

$$\begin{aligned}(1+2x^{-\frac{1}{2}})^{-2} &= [1+(2x^{-\frac{1}{2}})]^{-2}\\ &= 1^{-2} - 2\cdot 1^{-3}\cdot(2x^{-\frac{1}{2}}) + 3\cdot 1^{-4}\cdot(2x^{-\frac{1}{2}})^2\\ &\qquad - 4\cdot 1^{-5}\cdot(2x^{-\frac{1}{2}})^3 + 5\cdot 1^{-6}\cdot(2x^{-\frac{1}{2}})^4 - \cdots\\ &= 1 - 4x^{-\frac{1}{2}} + 12x^{-1} - 32x^{-\frac{3}{2}} + 80x^{-2} + \cdots.\end{aligned}$$

By writing the exponents of 1, in expanding $[1+(2x^{-\frac{1}{2}})]^{-2}$, we can make use of the fifth law of § 386.

3. Expand $\dfrac{1}{\sqrt[3]{a^{-1}-3x^{\frac{1}{3}}}}$ to four terms.

Enclosing a^{-1} and $-3x^{\frac{1}{3}}$ in parentheses, we have

$$\begin{aligned}\frac{1}{\sqrt[3]{a^{-1}-3x^{\frac{1}{3}}}} &= \frac{1}{(a^{-1}-3x^{\frac{1}{3}})^{\frac{1}{3}}} = [(a^{-1})+(-3x^{\frac{1}{3}})]^{-\frac{1}{3}}\\ &= (a^{-1})^{-\frac{1}{3}} - \frac{1}{3}(a^{-1})^{-\frac{4}{3}}(-3x^{\frac{1}{3}}) + \frac{2}{9}(a^{-1})^{-\frac{7}{3}}(-3x^{\frac{1}{3}})^2\\ &\qquad - \frac{14}{81}(a^{-1})^{-\frac{10}{3}}(-3x^{\frac{1}{3}})^3 + \cdots\\ &= a^{\frac{1}{3}} + a^{\frac{4}{3}}x^{\frac{1}{3}} + 2a^{\frac{7}{3}}x^{\frac{2}{3}} + \frac{14}{3}a^{\frac{10}{3}}x + \cdots.\end{aligned}$$

EXERCISE 176

Expand each of the following to five terms:

1. $(a+x)^{\frac{3}{2}}$.
2. $(1+x)^{-3}$.
3. $(1-x)^{-\frac{3}{4}}$.
4. $\sqrt[5]{a-b}$.
5. $\dfrac{1}{(a+x)^5}$.
6. $\dfrac{1}{\sqrt[6]{1-x}}$.
7. $(a^{\frac{4}{3}}+2b)^{\frac{3}{4}}$.
8. $(a^3-4x^{\frac{1}{2}})^{-\frac{7}{2}}$.
9. $\dfrac{1}{x^{-\frac{2}{3}}+3y}$.
10. $\left(m^{-3}+\dfrac{n^{-\frac{5}{6}}}{4}\right)^{-3}$.
11. $\sqrt[3]{[(a^{-2}-6b^2c)^7]}$.
12. $\dfrac{1}{(x^{-\frac{1}{3}}-2y^{\frac{3}{2}})^4}$.
13. $\left(\dfrac{x^2}{y}+\dfrac{y^2}{x}\right)^{-\frac{1}{5}}$.
14. $(m^{\frac{5}{3}}-3n^{-\frac{4}{3}})^{-\frac{4}{3}}$.
15. $\left(\dfrac{1}{5\sqrt[5]{a^4}}-\sqrt[3]{b^2}\right)^{\frac{3}{5}}$.

409. The formula for the rth term of $(a+x)^n$ (§ 390) holds for fractional or negative values of n, since it was derived from an expansion which holds for all values of the exponent.

Ex. Find the 7th term of $(a - 3x^{-\frac{3}{2}})^{-\frac{1}{3}}$.

Enclosing $-3x^{-\frac{3}{2}}$ in parentheses, we have

$$(a - 3x^{-\frac{3}{2}})^{-\frac{1}{3}} = [a + (-3x^{-\frac{3}{2}})]^{-\frac{1}{3}}.$$

The exponent of $(-3x^{-\frac{3}{2}})$ is $7-1$, or 6.

The exponent of a is $-\frac{1}{3} - 6$, or $-\frac{19}{3}$.

The first factor of the numerator is $-\frac{1}{3}$, and the last factor $-\frac{19}{3}+1$, or $-\frac{16}{3}$.

The last factor of the denominator is 6.

Hence, the 7th term

$$= \frac{-\frac{1}{3}\cdot -\frac{4}{3}\cdot -\frac{7}{3}\cdot -\frac{10}{3}\cdot -\frac{13}{3}\cdot -\frac{16}{3}}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6} a^{-\frac{19}{3}}(-3x^{-\frac{3}{2}})^6$$

$$= \frac{728}{3^8} a^{-\frac{19}{3}}(3^6 x^{-9}) = \frac{728}{9} a^{-\frac{19}{3}} x^{-9}.$$

EXERCISE 177

Find the:

1. 6th term of $(a+x)^{\frac{2}{3}}$.
2. 5th term of $(a-b)^{-\frac{1}{5}}$.
3. 7th term of $(1+x)^{-7}$.
4. 8th term of $(1-x)^{\frac{3}{5}}$.
5. 9th term of $(a-x)^{-8}$.
6. 11th term of $\sqrt{(m+n)^5}$.
7. 7th term of $(a^{-2} - 2b^{\frac{1}{3}})^{-2}$.
8. 8th term of $\dfrac{1}{(x^5 + y^{-\frac{1}{2}})^4}$.
9. 10th term of $(x^{-5} + y^{\frac{2}{3}})^{-\frac{5}{2}}$.
10. 6th term of $(a^{\frac{3}{2}} - 2b^{-4})^{-\frac{3}{4}}$.
11. 5th term of $(m + 3n^{-3})^{\frac{3}{5}}$.
12. 9th term of $\dfrac{1}{\sqrt[3]{[(a^3 + 3b^{-\frac{2}{3}})^5]}}$.

13. 11th term of $\left(a\sqrt[3]{b^4}-\dfrac{1}{\sqrt{c^5}}\right)^{-\frac{3}{2}}$.

14. 10th term of $(x^{-\frac{1}{3}}-4\,y^{\frac{3}{4}})^{\frac{9}{4}}$.

410. Extraction of Roots.

The Binomial Theorem may sometimes be used to find the approximate root of a number which is not a perfect power of the same degree as the index of the root.

Ex. **Find $\sqrt[3]{25}$ approximately to five places of decimals.**

The nearest perfect cube to 25 is 27.

We have $\sqrt[3]{25}=\sqrt[3]{27-2}=[(3^3)+(-2)]^{\frac{1}{3}}$

$$=(3^3)^{\frac{1}{3}}+\frac{1}{3}(3^3)^{-\frac{2}{3}}(-2)-\frac{1}{9}(3^3)^{-\frac{5}{3}}(-2)^2$$

$$+\frac{5}{81}(3^3)^{-\frac{8}{3}}(-2)^3-\cdots$$

$$=3-\frac{2}{3\cdot 3^2}-\frac{4}{9\cdot 3^5}-\frac{40}{81\cdot 3^8}-\cdots.$$

Expressing each fraction approximately to the nearest fifth decimal place, we have

$$\sqrt[3]{25}=3-.07407-.00183-.00008-\cdots=2.92402.$$

We then have the following rule:

Separate the given number into two parts, the first of which is the nearest perfect power of the same degree as the required root, and expand the result by the Binomial Theorem.

If the ratio of the second term of the binomial to the first is a small proper fraction, the terms of the expansion diminish rapidly; but if this ratio is but little less than 1, it requires a great many terms to insure any degree of accuracy.

EXERCISE 178

Find the approximate values of the following to five places of decimals:

1. $\sqrt{17}$. **2.** $\sqrt{51}$. **3.** $\sqrt[3]{60}$. **4.** $\sqrt[4]{14}$. **5.** $\sqrt[4]{84}$. **6.** $\sqrt[5]{3}$

XXXI. LOGARITHMS

411. The Common System.

Every positive arithmetical number may be expressed, exactly or approximately, as a power of 10.

Thus, $100 = 10^2$; $13 = 10^{1.1139\cdots}$; etc.

When thus expressed, the corresponding exponent is called its **Logarithm to the Base 10.**

Thus, 2 is the logarithm of 100 to the base 10; a relation which is written $\log_{10} 100 = 2$, or simply $\log 100 = 2$.

Logarithms of numbers to the base 10 are called *Common Logarithms*, and, collectively, form the *Common System.*

They are the only ones used for numerical computations.

412. Any positive number, except unity, may be taken as the base of a system of logarithms; thus, if $a^x = m$, where a and m are positive numbers, then $x = \log_a m$.

A negative number is not considered as having a logarithm.

413. By §§ 238 and 239,

$$10^0 = 1, \qquad 10^{-1} = \frac{1}{10} = .1,$$

$$10^1 = 10, \qquad 10^{-2} = \frac{1}{10^2} = .01,$$

$$10^2 = 100, \qquad 10^{-3} = \frac{1}{10^3} = .001, \text{ etc.}$$

Whence, by the definition of § 411,

$$\log 1 = 0, \qquad \log .1 = -1 = 9 - 10,$$

$$\log 10 = 1, \qquad \log .01 = -2 = 8 - 10,$$

$$\log 100 = 2, \qquad \log .001 = -3 = 7 - 10, \text{ etc.}$$

The second form for log .1, log .01, etc., is preferable in practice.

If no base is expressed, the base 10 is understood.

414. It is evident from § 413 that the common logarithm of a number greater than 1 is positive, and the logarithm of a number between 0 and 1 negative.

415. If a number is not an exact power of 10, its common logarithm can only be expressed approximately; the integral part of the logarithm is called the *characteristic*, and the decimal part the *mantissa*.

For example, $\log 13 \doteq 1.1139$.

Here, the characteristic is 1, and the mantissa .1139.

A negative logarithm is always expressed with a positive mantissa, which is done by adding and subtracting 10.

Thus, the negative logarithm -2.5863 is written $7.4137-10$.

In this case, $7-10$ is the characteristic.

The negative logarithm $7.4137-10$ is sometimes written $\bar{3}.4137$; the negative sign over the characteristic showing that it alone is negative, the mantissa being always positive.

For reasons which will appear, only the mantissa of the logarithm is given in a table of logarithms of numbers; the characteristic must be found by aid of the rules of §§ 416 and 417.

416. It is evident from § 413 that the logarithm of a number between

1 and 10 is equal to $0+$ a decimal;
10 and 100 is equal to $1+$ a decimal;
100 and 1000 is equal to $2+$ a decimal; etc.

Therefore, the characteristic of the logarithm of a number with *one* place to the left of the decimal point is 0; with *two* places to the left of the decimal point is 1; with *three* places to the left of the decimal point is 2; etc.

Hence, *the characteristic of the logarithm of a number greater than* 1 *is* 1 *less than the number of places to the left of the decimal point.*

For example, the characteristic of log 906328.51 is 5.

417. In like manner, the logarithm of a number between

1 and .1 is equal to 9 + a decimal − 10;
.1 and .01 is equal to 8 + a decimal − 10;
.01 and .001 is equal to 7 + a decimal − 10; etc.

Therefore, the characteristic of the logarithm of a decimal with *no* ciphers between its decimal point and first significant figure is 9, with −10 after the mantissa; of a decimal with *one* cipher between its point and first significant figure is 8, with −10 after the mantissa; of a decimal with *two* ciphers between its point and first significant figure is 7, with −10 after the mantissa; etc.

Hence, *to find the characteristic of the logarithm of a number less than* 1, *subtract the number of ciphers between the decimal point and first significant figure from* 9, *writing* −10 *after the mantissa.*

For example, the characteristic of log .007023 is 7, with −10 written after the mantissa.

PROPERTIES OF LOGARITHMS

418. *In any system, the logarithm of* 1 *is* 0.

For by § 238, $a^0 = 1$; whence, by § 412, $\log_a 1 = 0$.

419. *In any system, the logarithm of the base is* 1.

For, $a^1 = a$; whence, $\log_a a = 1$.

420. *In any system whose base is greater than* 1, *the logarithm of* 0 *is* $-\infty$.

For if a is greater than 1, $a^{-\infty} = \frac{1}{a^\infty} = \frac{1}{\infty} = 0$ (§ 320).

Whence, by § 412, $\log_a 0 = -\infty$.

No literal meaning can be attached to such a result as $\log_a 0 = -\infty$; it must be interpreted as follows:

If, in any system whose base is greater than unity, a number approaches the limit 0, its logarithm is negative, and increases indefinitely in absolute value. (Compare § 321.)

421. *In any system, the logarithm of a product is equal to the sum of the logarithms of its factors.*

Assume the equations

$$\left.\begin{array}{l} a^x = m \\ a^y = n \end{array}\right\}; \text{ whence, by § 412, } \left\{\begin{array}{l} x = \log_a m, \\ y = \log_a n. \end{array}\right.$$

Multiplying the assumed equations,

$$a^x \times a^y = mn, \text{ or } a^{x+y} = mn.$$

Whence, $\log_a mn = x + y = \log_a m + \log_a n.$

In like manner, the theorem may be proved for the product of three or more factors.

By aid of § 421, the logarithm of a composite number may be found when the logarithms of its factors are known.

Ex. Given $\log 2 = .3010$, and $\log 3 = .4771$; find $\log 72$.

$$\begin{aligned} \log 72 &= \log (2 \times 2 \times 2 \times 3 \times 3) \\ &= \log 2 + \log 2 + \log 2 + \log 3 + \log 3 \\ &= 3 \times \log 2 + 2 \times \log 3 = .9030 + .9542 = 1.8572. \end{aligned}$$

EXERCISE 179

Given

$\log 2 = .3010$, $\log 3 = .4771$, $\log 5 = .6990$, $\log 7 = .8451$, find:

1. log 15.	**4.** log 125.	**7.** log 567.	**10.** log 1875.
2. log 98.	**5.** log 315.	**8.** log 1225.	**11.** log 2646.
3. log 84.	**6.** log 392.	**9.** log 1372.	**12.** log 24696.

422. *In any system, the logarithm of a fraction is equal to the logarithm of the numerator minus the logarithm of the denominator.*

Assume the equations

$$\left.\begin{array}{l} a^x = m \\ a^y = n \end{array}\right\}; \text{ whence, } \left\{\begin{array}{l} x = \log_a m, \\ y = \log_a n. \end{array}\right.$$

Dividing the assumed equations,

$$\frac{a^x}{a^y} = \frac{m}{n}, \text{ or } a^{x-y} = \frac{m}{n}.$$

Whence, $$\log_a \frac{m}{n} = x - y = \log_a m - \log_a n.$$

Ex. Given $\log 2 = .3010$; find $\log 5$.

$$\log 5 = \log \frac{10}{2} = \log 10 - \log 2 = 1 - .3010 = .6990.$$

EXERCISE 180

Given $\log 2 = .3010$, $\log 3 = .4771$, $\log 7 = .8451$, find:

1. $\log \frac{10}{7}$.
2. $\log \frac{27}{2}$.
3. $\log 11\frac{1}{9}$.
4. $\log 245$.
5. $\log 85\frac{3}{4}$.
6. $\log 175$.
7. $\log \frac{49}{48}$.
8. $\log 375$.
9. $\log \frac{54}{25}$.
10. $\log \frac{300}{49}$.
11. $\log 46\frac{2}{7}$.
12. $\log 2\frac{11}{35}$.

423. *In any system, the logarithm of any power of a number is equal to the logarithm of the number multiplied by the exponent of the power.*

Assume the equation $a^x = m$; whence, $x = \log_a m$.

Raising both members of the assumed equation to the pth power, $$a^{px} = m^p; \text{ whence, } \log_a m^p = px = p \log_a m.$$

424. *In any system, the logarithm of any root of a number is equal to the logarithm of the number divided by the index of the root.*

For, $\log_a \sqrt[r]{m} = \log_a (m^{\frac{1}{r}}) = \frac{1}{r} \log_a m$ (§ 423).

425. **Examples.**

1. Given $\log 2 = .3010$; find $\log 2^{\frac{5}{3}}$.

$$\log 2^{\frac{5}{3}} = \frac{5}{3} \times \log 2 = \frac{5}{3} \times .3010 = .5017.$$

To multiply a logarithm by a fraction, multiply first by the numerator, and divide the result by the denominator.

2. Given $\log 3 = .4771$; find $\log \sqrt[8]{3}$.

$$\log \sqrt[8]{3} = \frac{\log 3}{8} = \frac{.4771}{8} = .0596.$$

3. Given $\log 2 = .3010$, $\log 3 = .4771$, find $\log (2^{\frac{1}{3}} \times 3^{\frac{5}{4}})$.

By § 421, $\log (2^{\frac{1}{3}} \times 3^{\frac{5}{4}}) = \log 2^{\frac{1}{3}} + \log 3^{\frac{5}{4}}$

$$= \frac{1}{3} \log 2 + \frac{5}{4} \log 3 = .1003 + .5964 = .6967.$$

EXERCISE 181

Given $\log 2 = .3010$, $\log 3 = .4771$, $\log 7 = .8451$, find:

1. $\log 2^8$.
2. $\log 5^7$.
3. $\log 3^{\frac{4}{3}}$.
4. $\log 7^{\frac{2}{5}}$.
5. $\log 42^6$.
6. $\log 45^{\frac{1}{4}}$.
7. $\log 63^{\frac{1}{6}}$.
8. $\log 98^{\frac{7}{2}}$.
9. $\log 50^{\frac{3}{7}}$.
10. $\log \sqrt[5]{3}$.
11. $\log \sqrt[8]{5}$.
12. $\log \sqrt[13]{7}$.
13. $\log \sqrt[7]{8}$.
14. $\log \sqrt[4]{54}$.
15. $\log \sqrt[6]{225}$.
16. $\log \sqrt[9]{162}$.
17. $\log \sqrt[12]{\frac{7}{3}}$.
18. $\log (\frac{5}{2})^{\frac{1}{5}}$.
19. $\log (3^{\frac{3}{2}} \times 100^{\frac{1}{3}})$.
20. $\log (5\sqrt[11]{3})$.
21. $\log \dfrac{\sqrt[7]{7}}{\sqrt[8]{2}}$.
22. $\log \dfrac{2^{\frac{2}{5}}}{5^{\frac{1}{6}}}$.
23. $\log \dfrac{\sqrt[4]{35}}{7^{\frac{3}{8}}}$.
24. $\log \dfrac{3^{\frac{4}{7}}}{\sqrt[9]{75}}$.

426. *To prove the relation*

$$\log_b m = \frac{\log_a m}{\log_a b}.$$

Assume the equations

$$\left.\begin{matrix} a^x = m \\ b^y = m \end{matrix}\right\}; \quad \text{whence,} \quad \left\{\begin{matrix} x = \log_a m, \\ y = \log_b m. \end{matrix}\right.$$

From the assumed equations, $a^x = b^y$.

Taking the yth root of both members, $a^{\frac{x}{y}} = b$.

Therefore, $\log_a b = \dfrac{x}{y}$, or $y = \dfrac{x}{\log_a b}$.

That is, $\log_b m = \dfrac{\log_a m}{\log_a b}$.

By aid of this relation, if the logarithm of a number m to a certain base a is known, its logarithm to any other base b may be found by dividing by the logarithm of b to the base a.

427. *To prove the relation*

$$\log_b a \times \log_a b = 1.$$

Putting $m = a$ in the result of § 426,

$$\log_b a = \frac{\log_a a}{\log_a b} = \frac{1}{\log_a b} \text{ (§ 419).}$$

Whence, $$\log_b a \times \log_a b = 1.$$

428. *In the common system, the mantissæ of the logarithms of numbers having the same sequence of figures are equal.*

Suppose, for example, that $\log 3.053 = .4847$.

Then, $\log 305.3 = \log(100 \times 3.053) = \log 100 + \log 3.053$
$= 2 + .4847 = 2.4847;$

$\log .03053 = \log(.01 \times 3.053) = \log .01 + \log 3.053$
$= 8 - 10 + .4847 = 8.4847 - 10;$ etc.

It is evident from the above that, if a number be multiplied or divided by any integral power of 10, producing another number with the same sequence of figures, the mantissæ of their logarithms will be equal.

For this reason, only mantissæ are given, in a table of Common Logarithms; for to find the logarithm of any number, we have only to find the mantissa corresponding to its sequence of figures, and then prefix the characteristic in accordance with the rules of §§ 416 and 417.

This property of logarithms only holds for the common system, and constitutes its superiority over other systems for numerical computation.

429. *Ex.* Given $\log 2 = .3010$, $\log 3 = .4771$; find $\log .00432$.

We have $\log 432 = \log(2^4 \times 3^3) = 4 \log 2 + 3 \log 3 = 2.6323.$

Then, by § 428, the *mantissa* of the result is .6353.
Whence, by § 417, $\log .00432 = 7.6353 - 10$.

EXERCISE 182

Given $\log 2 = .3010$, $\log 3 = .4771$, $\log 7 = .8451$, find:

1. $\log 2.7$.	6. $\log .00000686$.	11. $\log 337.5$.
2. $\log 14.7$.	7. $\log .00125$.	12. $\log 3.888$.
3. $\log .56$.	8. $\log 5670$.	13. $\log (4.5)^8$.
4. $\log .0162$.	9. $\log .0000588$.	14. $\log \sqrt[6]{8.4}$.
5. $\log 22.5$.	10. $\log .000864$.	15. $\log (24.3)^{\frac{2}{5}}$.

USE OF THE TABLE

430. The table (pages 384 and 385) gives the mantissæ of the logarithms of all integers from 100 to 1000, calculated to four places of decimals.

431. *To find the logarithm of a number of three figures.*

Look in the column headed "No." for the first two significant figures of the given number.

Then the required mantissa will be found in the corresponding horizontal line, in the vertical column headed by the third figure of the number.

Finally, prefix the characteristic in accordance with the rules of §§ 416 and 417.

For example, $\log 168 = 2.2253$;

$$\log .344 = 9.5366 - 10; \text{ etc.}$$

For a number consisting of one or two significant figures, the column headed 0 may be used.

Thus, let it be required to find $\log 83$ and $\log 9$.

By § 428, $\log 83$ has the same mantissa as $\log 830$, and $\log 9$ the same mantissa as $\log 900$.

Hence, $\log 83 = 1.9191$, and $\log 9 = 0.9542$.

No.	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396
No.	0	1	2	3	4	5	6	7	8	9

No.	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996
No.	0	1	2	3	4	5	6	7	8	9

432. *To find the logarithm of a number of more than three figures.*

1. Required the logarithm of 327.6.

We find from the table, $\log 327 = 2.5145,$

$\log 328 = 2.5159.$

That is, an increase of one unit in the number produces an increase of .0014 in the logarithm.

Then an increase of .6 of a unit in the number will increase the logarithm by $.6 \times .0014$, or .0008 to the nearest fourth decimal place.

Whence, $\log 327.6 = 2.5145 + .0008 = 2.5153.$

In finding the logarithm of a number, the difference between the next less and next greater mantissæ is called the *tabular difference;* thus, in Ex. 1, the tabular difference is .0014.

The subtraction may be performed mentally.

The following rule is derived from the above:

Find from the table the mantissa of the first three significant figures, and the tabular difference.

Multiply the latter by the remaining figures of the number, with a decimal point before them.

Add the result to the mantissa of the first three figures, and prefix the proper characteristic.

In finding the correction to the nearest units' figure, the decimal portion should be omitted, provided that if it is .5, or greater than .5, the units' figure is increased by 1; thus, 13.26 would be taken as 13, 30.5 as 31, and 22.803 as 23.

2. Find the logarithm of .021508.

Mantissa 215 =	.3324	Tab. diff. =	21
	2		.08
	.3326	Correction =	1.68 = 2, nearly.

The result is $8.3326 - 10$.

EXERCISE 183

Find the logarithms of the following:

1. 64. **2.** 3.7 **3.** 982. **4.** .798

5. 1079.	9. .00005023.	13. 7.3165.
6. .6757.	10. .0002625.	14. .019608.
7. .09496.	11. 31.393.	15. 810.39.
8. 4.288.	12. 48387.	16. .0025446.

433. *To find the number corresponding to a logarithm.*

1. Required the number whose logarithm is 1.6571.

Find in the table the mantissa 6571.

In the corresponding line, in the column headed "No.," we find 45, the first two figures of the required number, and at the head of the column we find 4, the third figure.

Since the characteristic is 1, there must be two places to the left of the decimal point (§ 416).

Hence, the number corresponding to 1.6571 is 45.4.

2. Required the number whose logarithm is 2.3934.

We find in the table the mantissæ 3927 and 3945.

The numbers corresponding to the logarithms 2.3927 and 2.3945 are 247 and 248, respectively.

That is, an increase of .0018 in the mantissa produces an increase of one unit in the number corresponding.

Then, an increase of .0007 in the mantissa will increase the number by $\frac{7}{18}$ of a unit, or .4, nearly.

Hence, the number corresponding is 247 + .4, or 247.4.

The following rule is derived from the above:

Find from the table the next less mantissa, the three figures corresponding, and the tabular difference.

Subtract the next less from the given mantissa, and divide the remainder by the tabular difference.

Annex the quotient to the first three figures of the number, and point off the result.

The rules for pointing off are the reverse of those of §§ 416 and 417:

I. *If* − 10 *is not written after the mantissa, add* 1 *to the characteristic, giving the number of places to the left of the decimal point.*

II. *If* − 10 *is written after the mantissa, subtract the positive part of the characteristic from* 9, *giving the number of ciphers to be placed between the decimal point and first significant figure.*

3. Find the number whose logarithm is $8.5265 - 10$.

$$\begin{array}{l} \quad\quad\quad\quad\quad\quad\quad\quad 5265 \\ \text{Next less mant.} = \underline{5263}\text{; figures corresponding, 336.} \\ \quad\quad\text{Tab. diff. } 13)2.00(.15 = .2\text{, nearly.} \\ \quad\quad\quad\quad\quad\quad\quad\quad\quad \underline{1\,3} \\ \quad\quad\quad\quad\quad\quad\quad\quad\quad\ 70 \end{array}$$

By the above rule, there will be one cipher to be placed between the decimal point and first significant figure; the result is .03362.

The correction can usually be depended upon to only one decimal place; the division should be carried to two places to determine the last figure accurately.

EXERCISE 184

Find the numbers corresponding to the following:

1. 0.8189.
2. $7.6064 - 10$.
3. 1.8767.
4. 2.6760.
5. 3.9826.
6. $8.7954 - 10$.
7. $6.5993 - 10$.
8. $9.9437 - 10$.
9. 0.7781.
10. $5.4571 - 10$.
11. 1.3019.
12. $4.2527 - 10$.
13. 2.0159.
14. $3.7264 - 10$.
15. 4.4929.

APPLICATIONS

434. The approximate value of a number in which the operations indicated involve only multiplication, division, involution, or evolution may be conveniently found by logarithms.

The utility of the process consists in the fact that addition takes the place of multiplication, subtraction of division, multiplication of involution, and division of evolution.

1. Find the value of $.0631 \times 7.208 \times .51272$.

By § 421, $\log (.0631 \times 7.208 \times .51272)$

$$= \log .0631 + \log 7.208 + \log .51272.$$

$$\begin{aligned} \log\ .0631 &= 8.8000 - 10 \\ \log\ 7.208 &= 0.8578 \\ \log .51272 &= \underline{9.7099 - 10} \end{aligned}$$

Adding, $\quad \log \text{ of result} = 19.3677 - 20 = 9.3677 - 10$ (See Note 1.)

Number corresponding to $9.3677 - 10 = .2332$.

Note 1. If the sum is a negative logarithm, it should be written in such a form that the negative portion of the characteristic may be -10.

Thus, $19.3677-20$ is written $9.3677-10$.

(In computations with four-place logarithms, the result cannot usually be depended upon to more than *four* significant figures.)

2. Find the value of $\frac{336.8}{7984}$.

By § 422, $$\log\frac{336.8}{7984}=\log 336.8-\log 7984.$$

$$\begin{aligned}\log 336.8 &= 12.5273-10\\ \log 7984 &= \underline{3.9022}\\ \text{Subtracting,}\quad \text{log of result} &= 8.6251-10 \qquad \text{(See Note 2.)}\end{aligned}$$

Number corresponding $=.04218$.

Note 2. To subtract a greater logarithm from a less, or a negative logarithm from a positive, increase the characteristic of the minuend by 10, writing -10 after the mantissa to compensate.

Thus, to subtract 3.9022 from 2.5273, write the minuend in the form $12.5273-10$; subtracting 3.9022 from this, the result is $8.6251-10$.

3. Find the value of $(.07396)^5$.

By § 423, $$\log(.07396)^5 = 5\times\log .07396.$$

$$\begin{aligned}\log .07396 &= 8.8690-10\\ &\ \underline{5}\\ &\ 44.3450-50\\ &= 4.3450-10=\log .000002213.\end{aligned}$$

4. Find the value of $\sqrt[3]{.035063}$.

By § 424, $$\log\sqrt[3]{.035063}=\frac{1}{3}\log .035063.$$

$$\log .035063 = 8.5449-10$$

$$3\overline{)28.5449-30}\quad\text{(See Note 3.)}$$
$$9.5150-10=\log .3274.$$

Note 3. To divide a negative logarithm, write it in such a form that the negative portion of the characteristic may be exactly divisible by the divisor, with -10 as the quotient.

Thus, to divide $8.5449-10$ by 3, we write the logarithm in the form $28.5449-30$; dividing this by 3, the quotient is $9.5150-10$.

EXERCISE 185

A *negative* number has no common logarithm (§ 412); if such numbers occur in computation, they may be treated as if they were positive, and the *sign* of the result determined irrespective of the logarithmic work.

Thus, in Ex. 3 of the following set, to find the value of $(-95.86) \times 3.3918$ we find the value of 95.86×3.3918, and put a $-$ sign before the result.

Find by logarithms the values of the following:

1. 4.253×7.104.
2. $6823.2 \times .1634$.
3. $(-95.86) \times 3.3918$.
4. $54.029 \times (-.0081487)$.
5. $.040764 \times .12896$.
6. $(-285.46) \times (-.00070682)$.
7. $\dfrac{5978}{9.762}$.
8. $\dfrac{21.658}{45057}$.
9. $\dfrac{.06405}{.002037}$.
10. $\dfrac{-38.19}{.10792}$.
11. $\dfrac{670.43}{-5382.3}$.
12. $\dfrac{.000007913}{.00082375}$.
13. $(88.08)^3$.
14. $(.09437)^4$.
15. $(3.625)^7$.
16. $(-.4623)^5$.
17. $100^{\frac{4}{5}}$.
18. $(.09)^{\frac{5}{2}}$.
19. $(85.7)^{\frac{3}{4}}$.
20. $(-.000216)^{\frac{6}{5}}$.
21. $\sqrt{7}$.
22. $\sqrt[4]{3}$.
23. $\sqrt[7]{-8}$.
24. $\sqrt[9]{10}$.
25. $\sqrt[5]{.2005}$.
26. $\sqrt[8]{.08367}$.
27. $\sqrt[6]{.00015027}$.
28. $\sqrt[11]{-.0040628}$.

435. Arithmetical Complement.

The *Arithmetical Complement* of the logarithm of a number, or, briefly, the *Cologarithm* of the number, is the logarithm of the reciprocal of that number.

Thus, $\quad \text{colog } 409 = \log \frac{1}{409} = \log 1 - \log 409.$

$$\begin{aligned} \log 1 &= 10. \qquad -10 \quad \text{(See Ex. 2, § 434.)} \\ \log 409 &= \underline{2.6117} \\ \therefore \text{colog } 409 &= 7.3883 - 10. \end{aligned}$$

Again, $\quad \text{colog } .067 = \log \dfrac{1}{.067} = \log 1 - \log .067.$

$$\begin{aligned}\log 1 &= 10. \qquad -10\\ \log .067 &= \underline{\ 8.8261-10\ }\\ \therefore\ \text{colog}\ .067 &= \ 1.1739.\end{aligned}$$

It follows from the above that *the cologarithm of a number may be found by subtracting its logarithm from* $10-10$.

The cologarithm may be found by subtracting the last *significant* figure of the logarithm from 10 and each of the others from 9, -10 being written after the result in the case of a positive logarithm.

Ex. Find the value of $\dfrac{.51384}{8.708 \times .0946}$.

$$\begin{aligned}\log \frac{.51384}{8.708 \times .0946} &= \log\left(.51384 \times \frac{1}{8.708} \times \frac{1}{.0946}\right)\\ &= \log .51384 + \log\frac{1}{8.708} + \log\frac{1}{.0946}\\ &= \log .51384 + \text{colog}\ 8.708 + \text{colog}\ .0946.\end{aligned}$$

$$\begin{aligned}\log .51384 &= 9.7109-10\\ \text{colog}\ 8.708 &= 9.0601-10\\ \text{colog}\ .0946 &= \underline{1.0241\qquad\quad}\\ & \ 9.7951-10 = \log .6239.\end{aligned}$$

It is evident from the above example that, to find the logarithm of a fraction whose terms are the products of factors, we *add together the logarithms of the factors of the numerator, and the cologarithms of the factors of the denominator.*

The value of the above fraction may be found without using cologarithms, by the following formula:

$$\begin{aligned}\log \frac{.51384}{8.709 \times .0946} &= \log .51384 - \log(8.709 \times .0946)\\ &= \log .51384 - (\log 8.709 + \log .0946).\end{aligned}$$

The advantage in the use of cologarithms is that the written work of computation is exhibited in a more compact form.

MISCELLANEOUS EXAMPLES

436. 1. Find the value of $\dfrac{2\sqrt[3]{5}}{3^{\frac{5}{8}}}$.

$$\log\frac{2\sqrt[3]{5}}{3^{\frac{5}{6}}} = \log 2 + \log\sqrt[3]{5} + \operatorname{colog} 3^{\frac{5}{6}} \quad (\S\ 435)$$
$$= \log 2 + \frac{1}{3}\log 5 + \frac{5}{6}\operatorname{colog} 3.$$

$$\begin{array}{llrl} & & \log 2 = & .3010 \\ \log 5 = & .6990; & +3 = & .2330 \\ \operatorname{colog} 3 = & 9.5229 - 10; & \times\frac{5}{6} = & \underline{9.6024 - 10} \\ & & & .1364 \quad = \log 1.369. \end{array}$$

2. Find the value of $\sqrt[3]{\dfrac{-.03296}{7.962}}$.

$$\log\sqrt[3]{\frac{.03296}{7.962}} = \frac{1}{3}\log\frac{.03296}{7.962} = \frac{1}{3}(\log .03296 - \log 7.962).$$

$$\begin{array}{rl} \log .03296 = & 8.5180 - 10 \\ \log\ \ 7.962 = & \underline{0.9010} \\ 3\,)\!\! & \underline{27.6170 - 30} \\ & 9.2057 - 10 = \log .1606. \end{array}$$

The result is $-.1606$.

EXERCISE 186

Find by logarithms the values of the following:

1. $\dfrac{2078.5 \times .05834}{.3583 \times 346}$.

2. $\dfrac{(-6.08) \times .1304}{4.046 \times .0031095}$.

3. $\dfrac{(-.076917) \times 26.3}{.5478 \times (-3120.7)}$.

4. $\dfrac{.8102 \times (-6.225)}{(-.0721) \times (-17.976)}$.

5. $6^{\frac{6}{5}} \times 5^{\frac{5}{6}}$.

6. $\dfrac{7^{\frac{4}{9}}}{9^{\frac{3}{8}}}$.

7. $\sqrt[11]{\dfrac{68}{35}}$.

8. $\dfrac{\sqrt[7]{8}}{(.1)^{\frac{2}{3}}}$.

9. $\dfrac{(100)^{\frac{2}{7}}}{\sqrt[3]{-.004}}$.

10. $\left(-\dfrac{5510}{7048}\right)^{\frac{4}{3}}$.

11. $\sqrt{\dfrac{38.7}{501.9}}$.

12. $\dfrac{\sqrt[5]{-.01}}{4^{\frac{9}{4}}}$.

13. $\dfrac{-(.03)^{\frac{5}{2}}}{\sqrt[9]{-1000}}$.

14. $\sqrt[4]{\dfrac{7}{9}} \div \sqrt[8]{\dfrac{3}{4}}$.

15. $\sqrt{6} \times \sqrt[6]{10} \times \sqrt[10]{2}$.

16. $\left(-\dfrac{24.18}{8.7 \times .0603}\right)^{\frac{3}{5}}$.

17. $\dfrac{\sqrt[5]{.008546}}{\sqrt[6]{.0003867}}$.

18. $\dfrac{(-.14582)^{\frac{5}{3}}}{-(.72346)^{\frac{7}{4}}}$.

19. $(-143.59)^{11} \times (.00532)^9$.

20. $\sqrt[9]{40.954 \times .0002098}$.

21. $(3075.6)^{\frac{1}{6}} \times (.016432)^{\frac{7}{4}}$.

22. $\dfrac{\sqrt[8]{28.18} \times \sqrt[8]{544.6}}{\sqrt[4]{61021}}$.

23. $\dfrac{(.0462)^{\frac{7}{5}}}{758.27 \times \sqrt{.2296}}$.

24. $\dfrac{\sqrt[3]{-7.92} \times (.1807)^{\frac{2}{3}}}{.0016445}$.

25. $\dfrac{-27.931}{\sqrt[4]{.836} \times (.03023)^{\frac{1}{7}}}$.

26. $\sqrt[5]{-.067268} \times \sqrt[7]{-.4175} \times \sqrt[9]{.00263}$.

27. $\dfrac{.0005616 \times \sqrt[7]{424.6}}{(6.73)^4 \times (.03194)^{\frac{5}{4}}}$.

28. $\dfrac{485.7 \times (.7301)^7 \times \sqrt[6]{1000}}{(9.127)^6 \times (.7095)^{\frac{2}{3}}}$.

EXPONENTIAL EQUATIONS

437. An **Exponential Equation** is an equation in which the unknown number occurs as an exponent.

To solve an equation of this form, take the logarithms of both members; the result will be an equation which can be solved by ordinary algebraic methods.

1. Given $31^x = 23$; find the value of x.

Taking the logarithms of both members,

$$\log(31^x) = \log 23; \text{ or } x \log 31 = \log 23 \ (\S\ 423).$$

Then,
$$x = \frac{\log 23}{\log 31} = \frac{1.3617}{1.4914} = .9130+.$$

2. Solve the equation $.2^x = 3$.

Taking the logarithms of both members, $x \log .2 = \log 3$.

Then,
$$x = \frac{\log 3}{\log .2} = \frac{.4771}{9.3010 - 10} = \frac{.4771}{-.699} = -.6285+.$$

An equation of the form $a^x = b$ may be solved by inspection if b can be expressed as an exact power of a.

3. Solve the equation $16^x = 128$.

We may write the equation $(2^4)^x = 2^7$, or $2^{4x} = 2^7$.

Then, by inspection, $4x = 7$; and $x = \dfrac{7}{4}$.

(If the equation were $16^x = \frac{1}{128}$, we could write it $(2^4)^x = \frac{1}{2^7} = 2^{-7}$; then $4x$ would equal -7, and $x = -\frac{7}{4}$.)

EXERCISE 187

Solve the following equations:

1. $13^x = 8$.
2. $.06^x = .9$.
3. $9.347^x = .0625$.
4. $.005038^x = 816.3$.
5. $3^{4x-1} = 4^{2x+3}$.
6. $7^{3x+2} = .8^x$.
7. $.2^{x+5} = .5^{x-4}$.
8. $16^x = 32$.
9. $32^x = \frac{1}{64}$.
10. $(\frac{1}{16})^x = 8$.
11. $(\frac{1}{9})^x = \frac{1}{27}$.
12. Given a, r, and l; derive the formula for n (§ 372).
13. Given a, r, and S; derive the formula for n.
14. Given a, l, and S; derive the formula for n.
15. Given r, l, and S; derive the formula for n.

438. 1. Find the logarithm of .3 to the base 7.

By § 426, $\log_7 .3 = \frac{\log_{10} .3}{\log_{10} 7} = \frac{9.4771 - 10}{.8451} = -\frac{.5229}{.8451} = -.6187+$.

Examples of this kind may be solved by inspection, if the number can be expressed as an exact power of the base.

2. Find the logarithm of 128 to the base 16.

Let $\log_{16} 128 = x$; then, by § 412, $16^x = 128$.

Then, as in Ex. 3, § 437, $x = \frac{7}{4}$; that is, $\log_{16} 128 = \frac{7}{4}$.

EXERCISE 188

Find the values of the following:

1. $\log_7 59$.
2. $\log_6 .7$.
3. $\log_{.4} 82$.
4. $\log_9 .00453$.
5. $\log_{68} 2.92$.
6. $\log_{21} .0604$.

Find by inspection the values of the following:

7. $\log_{25} 125$. 8. $\log_{49} (\frac{1}{7})$. 9. $\log_{\frac{1}{27}} (3)$. 10. $\log_{\frac{1}{32}} (\frac{1}{128})$.

XXXII. MISCELLANEOUS TOPICS

HIGHEST COMMON FACTOR AND LOWEST COMMON MULTIPLE BY DIVISION

439. We will now show how to find the H. C. F. of two polynomials which cannot be readily factored by inspection.

The rule in Arithmetic for the H. C. F. of two numbers is:

Divide the greater number by the less.

If there be a remainder, divide the divisor by it; and continue thus to make the remainder the divisor, and the preceding divisor the dividend, until there is no remainder.

The last divisor is the H. C. F. *required.*

Thus, let it be required to find the H. C. F. of 169 and 546.

$$\begin{array}{l} 169)546(3 \\ \quad\ \ \underline{507} \\ \quad\ \ \ 39)169(4 \\ \qquad\quad\ \ \underline{156} \\ \qquad\qquad 13)39(3 \\ \qquad\qquad\quad\ \ \underline{39} \end{array}$$

Then, 13 is the H. C. F. required.

440. We will now prove that a rule similar to that of § 439 holds for the H. C. F. of two algebraic expressions.

Let A and B be two polynomials, arranged according to the descending powers of some common letter.

Let the exponent of this letter in the first term of A be equal to, or greater than, its exponent in the first term of B.

Suppose that B is contained in A p times, with a remainder C; that C is contained in B q times, with a remainder D; and that D is contained in C r times, with no remainder.

To prove that D is the H. C. F. of A and B.

The operation of division is shown as follows.

$$\begin{array}{l} B)A(p \\ \underline{pB} \\ \quad C)B(q \\ \quad \underline{qC} \\ \qquad D)C(r \\ \qquad \underline{rD} \\ \qquad\ 0 \end{array}$$

We will first prove that D is a common factor of A and B.

Since the minuend is equal to the subtrahend plus the remainder (§ 34),

$$A = pB + C, \tag{1}$$

$$B = qC + D, \tag{2}$$

and

$$C = rD.$$

Substituting the value of C in (2), we obtain

$$B = qrD + D = D(qr + 1). \tag{3}$$

Substituting the values of B and C in (1), we have

$$A = pD(qr + 1) + rD = D(pqr + p + r). \tag{4}$$

From (3) and (4), D is a common factor of A and B.

We will next prove that every common factor of A and B is a factor of D.

Let F be any common factor of A and B; and let

$$A = mF, \text{ and } B = nF.$$

From the operation of division, we have

$$C = A - pB, \tag{5}$$

and

$$D = B - qC. \tag{6}$$

Substituting the values of A and B in (5), we have

$$C = mF - pnF.$$

Substituting the values of B and C in (6), we have

$$D = nF - q(mF - pnF) = F(n - qm + pqn).$$

Whence, F is a factor of D.

Then, since every common factor of A and B is a factor of D, and since D is itself a common factor of A and B, it follows that D is the *highest* common factor of A and B.

We then have the following rule for the H. C. F. of two polynomials, A and B, arranged according to the descending powers of some common letter, the exponent of that letter in the first term of A being equal to, or greater than, its exponent in the first term of B:

Divide A by B.

If there be a remainder, divide the divisor by it; and continue thus to make the remainder the divisor, and the preceding divisor the dividend, until there is no remainder.

The last divisor is the H. C. F. *required.*

It is important to keep the work throughout in descending powers of some common letter; and each division should be continued until the exponent of this letter in the first term of the remainder is less than its exponent in the first term of the divisor.

Note 1. If the terms of one of the expressions have a common factor which is not a common factor of the terms of the other, it may be removed; for it can evidently form no part of the highest common factor.

In like manner, we may divide any remainder by a factor which is not a factor of the preceding divisor.

1. Find the H. C. F. of

$$6x^2 - 25x + 14 \text{ and } 6x^3 - 7x^2 - 25x + 18.$$

$$\begin{array}{r}
6x^2 - 25x + 14)\,6x^3 - 7x^2 - 25x + 18\,(x + 3 \\
\underline{6x^3 - 25x^2 + 14x} \qquad\qquad \\
18x^2 - 39x + 18 \\
\underline{18x^2 - 75x + 42} \\
36x - 24
\end{array}$$

In accordance with Note 1, we divide this remainder by 12, giving $3x - 2$.

$$\begin{array}{r}
3x - 2)\,6x^2 - 25x + 14\,(2x - 7 \\
\underline{6x^2 - 4x} \qquad\qquad \\
-21x + 14 \\
\underline{-21x + 14}
\end{array}$$

Then, $3x - 2$ is the H. C. F. required.

Note 2. If the first term of the dividend, or of any remainder, is not divisible by the first term of the divisor, it may be made so by multiplying the dividend or remainder by any term which is not a factor of the divisor.

2. Find the H. C. F. of

$$3a^3 + a^2b - 2ab^2 \text{ and } 4a^3b + 2a^2b^2 - ab^3 + b^4.$$

We remove the factor a from the first expression and the factor b from the second (Note 1), and find the H. C. F. of

$$3a^2 + ab - 2b^2 \text{ and } 4a^3 + 2a^2b - ab^2 + b^3.$$

Since $4a^3$ is not divisible by $3a^2$, we multiply the second expression by 3 (Note 2).

$$\begin{array}{r|l}
 & 4a^3 + 2a^2b - ab^2 + b^3 \\
 & \underline{\qquad\qquad\qquad\qquad 3} \\
3a^2 + ab - 2b^2) & 12a^3 + 6a^2b - 3ab^2 + 3b^3(4a \\
 & \underline{12a^3 + 4a^2b - 8ab^2} \\
 & \quad 2a^2b + 5ab^2 + 3b^3
\end{array}$$

Since $2a^2b$ is not divisible by $3a^2$, we multiply this remainder by 3 (Note 2).

$$\begin{array}{r|l}
 & 2a^2b + 5ab^2 + 3b^3 \\
 & \underline{\qquad\qquad\qquad 3} \\
3a^2 + ab - 2b^2) & 6a^2b + 15ab^2 + 9b^3(2b \\
 & \underline{6a^2b + 2ab^2 - 4b^3} \\
 & \quad 13ab^2 + 13b^3
\end{array}$$

We divide this remainder by $13b^2$ (Note 1), giving $a + b$.

$$\begin{array}{r|l}
a + b) & 3a^2 + ab - 2b^2(3a - 2b \\
 & \underline{3a^2 + 3ab} \\
 & \quad -2ab \\
 & \quad \underline{-2ab - 2b^2}
\end{array}$$

Then, $a + b$ is the H. C. F. required.

Note 3. If the first term of any remainder is negative, the sign of each term of the remainder may be changed.

Note 4. If the given expressions have a common factor which can be seen by inspection, remove it, and find the H. C. F. of the resulting expressions; the result, multiplied by the common factor, will be the H. C. F. of the given expressions.

3. Find the H. C. F. of

$$2x^4+3x^3-6x^2+2x \text{ and } 6x^4+5x^3-2x^2-x.$$

Removing the common factor x (Note 4), we find the H. C. F. of

$$2x^3+3x^2-6x+2 \text{ and } 6x^3+5x^2-2x-1.$$

$$\begin{array}{r}
2x^3+3x^2-6x+2)\,6x^3+5x^2-\ 2x-1\,(3 \\
\underline{6x^3+9x^2-18x+6} \\
-4x^2+16x-7
\end{array}$$

The first term of this remainder being negative, we change the sign of each of its terms (Note 3).

$$\begin{array}{r}
2x^3+\ 3x^2-\ \ 6x+\ 2 \\
\underline{2} \\
4x^2-16x+7)\,4x^3+\ 6x^2-\ 12x+\ 4\,(x \\
\underline{4x^3-16x^2+\ \ 7x} \\
22x^2-\ 19x+\ 4 \\
\underline{2} \\
44x^2-\ 38x+\ 8\,(11 \\
\underline{44x^2-176x+77} \\
69)\,\underline{138x-69} \\
2x-\ 1 \\
2x-1)\,4x^2-\ 16x+\ 7\,(2x-7 \\
\underline{4x^2-\ \ 2x} \\
-\ 14x \\
\underline{-\ 14x+7}
\end{array}$$

The last divisor is $2x-1$; multiplying this by x, the H. C. F. of the given expressions is $x(2x-1)$.

(In the above solution, we multiply $2x^3+3x^2-6x+2$ by 2 in order to make its first term divisible by $4x^2$; and we multiply the remainder $22x^2-19x+4$ by 2 to make its first term divisible by $4x^2$.)

EXERCISE 189

Find the H. C. F. of the following:

1. $2a^2+a-6,\ 4a^2-8a+3.$
2. $6x^2-17x+10,\ 9x^2-14x-8.$
3. $x^2-6x-27,\ x^3-2x^2-8x+21.$

4. $6x^2 - x - 2,\ 3x^3 - 14x^2 - x + 6.$

5. $24a^2 - 22ab - 7b^2,\ 32a^2 - 12ab - 5b^2.$

6. $16x^3 + 8x^2y + 13xy^2 + 3y^3,\ 24x^3 - 14x^2y + 13xy^2 - 15y^3.$

7. $4x^3 + 4x^2 - 3x,\ 6x^4 + 11x^3 - x^2 - 6x.$

8. $4x^2y - 15xy^2 + 9y^3,\ 8x^4 - 18x^3y + 25x^2y^2 - 12xy^3.$

9. $6a^6 + 5a^5 - 6a^4 - 3a^3 + 2a^2,\ 9a^4 + 18a^3 + 5a^2 - 8a - 4.$

10. $3a^4 - 13a^3b + 3a^2b^2 + 4ab^3,\ 9a^3b + 12a^2b^2 - 8ab^3 - 5b^4.$

11. $4x^2 + 9x - 9,\ 4x^4 + 10x^3 - 7x^2 + 9.$

12. $6a^4 - 7a^3 - 5a^2 + 5a - 3,\ 8a^4 - 6a^3 - 5a^2 - 9.$

13. $3n^3 + 8n^2x - 9nx^2 + 2x^3,$
$6n^4 + 23n^3x + 2n^2x^2 - 13nx^3 + 2x^4.$

14. $a^3 + 9a^2 + 13a - 15,\ a^5 + 9a^4 + 22a^3 + 9a^2 - 9a.$

15. $m^6 - 27m^3,\ m^6 + 4m^5 - 25m^4 + 12m^3.$

16. $9a^4 + 30a^3b - 21a^2b^2 + 12ab^3,$
$16a^3b + 60a^2b^2 - 20ab^3 - 16b^4.$

17. $4x^2 - 11xy - 20y^2,\ 2x^4 - 4x^3y - 17x^2y^2 + xy^3 + 12y^4.$

18. $4a^5 + 8a^4 - 15a^3 + 2a^2 - 4a;\ 4a^4 - 12a^3 + 9a^2 - 3a + 2.$

19. $3x^3 - 8x^2 + 16x - 8,\ 3x^4 - 5x^3 + 5x^2 - 11x + 6.$

20. $3x^5y^2 - 2x^4y^3 - 7x^3y^4 + 7x^2y^5 + 3xy^6,$
$3x^5y^2 + 7x^4y^3 + 5x^3y^4 - 5x^2y^5 - 2xy^6.$

21. $2x^4 + 5x^3 + 4x^2 + 7x + 6,\ 2x^4 - 5x^3 + 11x^2 - 9x + 9.$

22. $6x^4 + x^3 + 3x^2 - 6x - 4,\ 12x^4 + 8x^3 - 3x^2 - 10x - 4.$

23. $3x^3 - 8x^2 - 5x + 6,\ x^5 - 5x^4 + 5x^3 + x^2 + 7x - 3.$

441. The H. C. F. of three expressions, which cannot be readily factored by inspection, may be found as follows:

Let A, B, and C be the expressions.

Let G be the H. C. F. of A and B; then, every common factor of G and C is a common factor of A, B, and C.

But since every common factor of two expressions exactly divides their H. C. F., every common factor of A, B, and C is also a common factor of G and C.

Whence, the H. C. F. of G and C is the H. C. F. of A, B, and C.

Hence, *to find the* H. C. F. *of three expressions, find the* H. C. F. *of two of them, and then of this result and the third expression.*

We proceed in a similar manner to find the H. C. F. of any number of expressions.

Ex. Find the H. C. F. of

x^3-7x+6, $x^3+3x^2-16x+12$, and x^3-5x^2+7x-3.

The H. C. F. of x^3-7x+6 and $x^3+3x^2-16x+12$ is x^2-3x+2.
The H. C. F. of x^2-3x+2 and x^3-5x^2+7x-3 is $x-1$.

EXERCISE 190

Find the H. C. F. of the following:

1. $6x^2-5x-25$, $9x^2+27x+20$, $12x^2+11x-15$.
2. $20a^2+23ab-7b^2$, $28a^2-43ab+9b^2$, $24a^2+14ab-5b^2$.
3. $5a^2-33a-14$, $5a^3-13a^2+14a+8$, $5a^3+27a^2+20a+4$.
4. $8x^2-6xy-35y^2$, $10x^3-27x^2y-xy^2+15y^3$, $6x^3-13x^2y-13xy^2+20y^3$.
5. $x^3-4x^2-11x+30$, x^3+2x^2-5x-6, $x^3-x^2-17x-15$.
6. $a^3-8a^2+20a-16$, $a^3+3a^2-4a-12$, $a^3-6a^2+11a-6$.
7. $3a^3+17a^2b+18ab^2-8b^3$, $6a^3+a^2b-19ab^2+6b^3$, $8a^3+6a^2b-23ab^2-6b^3$.
8. $3x^3-x^2-38x-24$, $3x^3+5x^2-58x-40$, $3x^3+26x^2+61x+30$.

442. We will now show how to find the L. C. M. of two expressions which cannot be readily factored by inspection.

Let A and B be any two expressions.

Let F be their H.C.F., and M their L.C.M.

Suppose that $A = aF$, and $B = bF$.

Then, $$A \times B = abF^2. \qquad (1)$$

Since F is the H.C.F. of A and B, a and b have no common factors; whence, the L.C.M. of aF and bF is abF.

That is, $$M = abF.$$

Multiplying each of these equals by F, we have

$$F \times M = abF^2. \qquad (2)$$

From (1) and (2), $$A \times B = F \times M. \qquad \text{(Ax. 4, § 9)}$$

That is, *the product of two expressions is equal to the product of their* H. C. F. *and* L. C. M.

Therefore, to find the L. C. M. of two expressions,

Divide their product by their highest common factor; or,

Divide one of the expressions by their highest common factor, and multiply the quotient by the other expression.

Ex. Find the L. C. M. of

$$6x^2 - 17x + 12 \text{ and } 12x^2 - 4x - 21.$$

$$\begin{array}{l}
6x^2 - 17x + 12)\,12x^2 - 4x - 21\,(2 \\
\qquad\qquad\qquad\;\; \underline{12x^2 - 34x + 24} \\
\qquad\qquad\qquad\qquad 15)\,30x - 45 \\
\qquad\qquad\qquad\qquad\quad 2x - 3)\,6x^2 - 17x + 12\,(3x - 4 \\
\qquad\qquad\qquad\qquad\qquad\qquad\;\; \underline{6x^2 - 9x} \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad - 8x \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{- 8x + 12}
\end{array}$$

Then, the H. C. F. of the expressions is $2x - 3$.

Dividing $6x^2 - 17x + 12$ by $2x - 3$, the quotient is $3x - 4$.

Then, the L. C. M. is $(3x - 4)(12x^2 - 4x - 21)$.

EXERCISE 191

Find the L. C. M. of the following:

1. $3x^2 + 14x - 24$, $3x^2 + 23x + 30$.

2. $6x^2-31xy+18y^2,\ 9x^2+15xy-14y^2.$
3. $4x^2+13x+3,\ 4x^2-23x-6.$
4. $8x^2+6x-9,\ 6x^3+7x^2-7x-6.$
5. $3a^3-8a^2b+4ab^2,\ a^3b-11a^2b^2+22ab^3-8b^4.$
6. $6n^3+n^2x-11nx^2-6x^3,\ 6n^3-5n^2x-8nx^2+3x^3.$
7. $2x^4+7x^3+7x^2+2x,\ 2x^4+x^3-10x^2-8x.$
8. $6x^3+x^2-17x+10,\ 3x^4+5x^3-5x^2-5x+2.$
9. $4x^2-11x-3,\ 8x^4+6x^3-11x^2-23x-5.$
10. $2x^4-x^3y-4x^2y^2+3xy^3,\ 8x^3y-10x^2y^2+12xy^3-10y^4.$
11. $6m^3-17m^2n-7mn^2+4n^3,\ 12m^3-13m^2n+21mn^2-6n^3.$
12. $2x^5+5x^4-2x^3+3x^2,\ 3x^6+8x^5-2x^4+x^3-6x^2.$
13. $a^4-2a^3-2a^2+7a-6,\ a^4-4a^3+a^2+7a-2.$

It follows from § 442 that, if two expressions are prime to each other (§ 128), their product is their L. C. M.

443. The L. C. M. of three expressions may be found as follows:

Let A, B, and C be the expressions.

Let M be the L. C. M. of A and B; then every common multiple of M and C is a common multiple of A, B, and C.

But since every common multiple of two expressions is exactly divisible by their L. C. M., every common multiple of A, B, and C is also a common multiple of M and C.

Then, the L. C. M. of M and C is the L. C. M. of A, B, and C.

Hence, *to find the* L. C. M. *of three expressions, find the* L. C. M. *of two of them, and then of this result and the third expression.*

We proceed in a similar manner to find the L. C. M. of any number of expressions.

EXERCISE 192

Find the L. C. M. of the following:

1. $3x^2-4x-4,\ 3x^2-7x+2,\ 3x^2-10x+8.$

2. $2a^5+3a^4-9a^3,\ 4a^4+13a^3+3a^2,\ 6a^3+13a^2-15a.$

3. $3n^2-11n-4,\ 4n^2-22n+24,\ 6n^2+11n+3.$

4. $4a^3+4a^2-43a+20,\ 4a^3+20a^2+13a-12,$
$4a^3+12a^2-31a-60.$

5. $2x^2-5x+3,\ 4x^3-4x^2+3x-9,\ 4x^3-13x+6.$

444. We will now show how to reduce a fraction to its lowest terms, when the numerator and denominator cannot be readily factored by inspection.

By § 127, the H. C. F. of two expressions is their common factor of highest degree, having the numerical coefficient of greatest absolute value in its term of highest degree.

We then have the following rule:

Divide both numerator and denominator by their H. C. F.

Ex. Reduce $\dfrac{6a^3-11a^2+7a-6}{2a^2-a-3}$ to its lowest terms.

By the rule of § 440, we find the H. C. F. of $6a^3-11a^2+7a-6$ and $2a^2-a-3$ to be $2a-3$.

Dividing $6a^3-11a^2+7a-6$ by $2a-3$, the quotient is $3a^2-a+2$.
Dividing $2a^2-a-3$ by $2a-3$, the quotient is $a+1$.

Then,
$$\frac{6a^3-11a^2+7a-6}{2a^2-a-3}=\frac{3a^2-a+2}{a+1}.$$

EXERCISE 193

Reduce each of the following to its lowest terms:

1. $\dfrac{3a^2-13a+12}{6a^2-5a-4}.$

2. $\dfrac{6m^2+7m-20}{4m^2+16m+15}.$

3. $\dfrac{5a^2-13ab-6b^2}{4a^2-15ab+9b^2}.$

4. $\dfrac{4x^2-4x-3}{4x^3-4x^2-5x-1}.$

5. $\dfrac{3x^4+7x^3-8x-8}{3x^2+10x+8}.$

6. $\dfrac{5a^3+11a^2+22a+4}{5a^3+8a^2+16a-8}.$

7. $\dfrac{4n^3-6n^2+8n-6}{n^3-3n^2-2n+4}.$

8. $\dfrac{4a^3+13a^2b-4ab^2-6b^3}{8a^2+14ab-15b^2}.$

9. $\dfrac{6x^3-x^2-11x+6}{9x^3-18x^2+11x-2}$.   10. $\dfrac{2x^3-9x^2y-2xy^2-15y^3}{2x^3-7x^2y-16xy^2+5y^3}$.

PROOF OF (1), § 235, FOR ALL VALUES OF m AND n

445. I. Let $m=\dfrac{p}{q}$ and $n=\dfrac{r}{s}$, where p, q, r, and s are positive integers.

We have, $a^{\frac{p}{q}}\times a^{\frac{r}{s}}=a^{\frac{ps}{qs}}\times a^{\frac{qr}{qs}}=\sqrt[qs]{a^{ps}}\times\sqrt[qs]{a^{qr}}$ (§ 237)

$=\sqrt[qs]{a^{ps}\times a^{qr}}$ (§ 234) $=\sqrt[qs]{a^{ps+qr}}$ (§ 56) $=a^{\frac{ps+qr}{qs}}$ (§ 237) $=a^{\frac{p}{q}+\frac{r}{s}}$.

We have now proved that (1), § 235, holds when m and n are any positive integers or positive fractions.

II. Let m be a positive integer or fraction; and let $n=-q$, where q is a positive integer or fraction less than m.

By §§ 56, or 445, I, $a^{m-q}\times a^q=a^{m-q+q}=a^m$.

Whence, $$a^{m-q}=\frac{a^m}{a^q}=a^m\times a^{-q}\ (\S\ 240).$$

That is, $$a^m\times a^{-q}=a^{m-q}.$$

III. Let m be a positive integer or fraction; and let $n=q$, where q is a positive integer or fraction greater than m.

By § 240, $a^m\times a^{-q}=\dfrac{1}{a^{-m}a^q}=\dfrac{1}{a^{-m+q}}$ (§ 445, II) $=a^{m-q}$.

IV. Let $m=-p$ and $n=-q$, where p and q are positive integers or fractions.

Then, $a^{-p}\times a^{-q}=\dfrac{1}{a^pa^q}=\dfrac{1}{a^{p+q}}$(§§ 56, or 445, I)$=a^{-p-q}$.

Then, $a^m\times a^n=a^{m+n}$ for all positive or negative, integral or fractional, values of m and n.

446. We will now show how to reduce a fraction whose denominator is irrational to an equivalent fraction having a rational denominator, when the denominator is the sum of a rational expression and a surd of the nth degree, or of two surds of the nth degree.

1. Reduce $\dfrac{1}{2+\sqrt[3]{3}}$ to an equivalent fraction having a rational denominator.

We have,
$$\frac{1}{2+\sqrt[3]{3}}=\frac{1}{8^{\frac{1}{3}}+3^{\frac{1}{3}}}.$$

Now, $(a+b)(a^2-ab+b^2)=a^3+b^3$ (§ 102).

Then, if we multiply both terms by $8^{\frac{2}{3}}-8^{\frac{1}{3}}\cdot 3^{\frac{1}{3}}+3^{\frac{2}{3}}$, the denominator will become rational; thus,

$$\frac{1}{8^{\frac{1}{3}}+3^{\frac{1}{3}}}=\frac{8^{\frac{2}{3}}-8^{\frac{1}{3}}\cdot 3^{\frac{1}{3}}+3^{\frac{2}{3}}}{(8^{\frac{1}{3}}+3^{\frac{1}{3}})(8^{\frac{2}{3}}-8^{\frac{1}{3}}\cdot 3^{\frac{1}{3}}+3^{\frac{2}{3}})}=\frac{(8^{\frac{1}{3}})^2-8^{\frac{1}{3}}\cdot 3^{\frac{1}{3}}+3^{\frac{2}{3}}}{8+3}$$
$$=\frac{4-2\sqrt[3]{3}+\sqrt[3]{9}}{11}.$$

2. Reduce $\dfrac{1}{\sqrt[4]{7}-\sqrt[4]{5}}$ to an equivalent fraction having a rational denominator.

We have,
$$\frac{1}{\sqrt[4]{7}-\sqrt[4]{5}}=\frac{1}{7^{\frac{1}{4}}-5^{\frac{1}{4}}}.$$

Now, $(a-b)(a^3+a^2b+ab^2+b^3)=a^4-b^4$ (§ 103).

Then, if we multiply both terms by
$$7^{\frac{3}{4}}+7^{\frac{2}{4}}\cdot 5^{\frac{1}{4}}+7^{\frac{1}{4}}\cdot 5^{\frac{2}{4}}+5^{\frac{3}{4}},$$
the denominator will become rational; thus,

$$\frac{1}{7^{\frac{1}{4}}-5^{\frac{1}{4}}}=\frac{7^{\frac{3}{4}}+7^{\frac{2}{4}}\cdot 5^{\frac{1}{4}}+7^{\frac{1}{4}}\cdot 5^{\frac{2}{4}}+5^{\frac{3}{4}}}{(7^{\frac{1}{4}}-5^{\frac{1}{4}})(7^{\frac{3}{4}}+7^{\frac{2}{4}}\cdot 5^{\frac{1}{4}}+7^{\frac{1}{4}}\cdot 5^{\frac{2}{4}}+5^{\frac{3}{4}})}$$
$$=\frac{\sqrt[4]{7^3}+\sqrt[4]{7^2\cdot 5}+\sqrt[4]{7\cdot 5^2}+\sqrt[4]{5^3}}{7-5}=\frac{\sqrt[4]{343}+\sqrt[4]{245}+\sqrt[4]{175}+\sqrt[4]{125}}{2}.$$

The method of § 446 can be applied to cases where the denominator is in the form $\sqrt[m]{a}+\sqrt[n]{b}$, or $\sqrt[m]{a}-\sqrt[n]{b}$.

3. Reduce $\dfrac{1}{\sqrt[3]{2}+\sqrt{5}}$ to an equivalent fraction having a rational denominator.

The *lowest common multiple of the indices* 3 *and* 2 *is* 6.

We have, $\dfrac{1}{\sqrt[3]{2}+\sqrt{5}}=\dfrac{1}{\sqrt[6]{2^2}+\sqrt[6]{5^3}}=\dfrac{1}{(2^2)^{\frac{1}{6}}+(5^3)^{\frac{1}{6}}}.$

Now, $(a+b)(a^5-a^4b+a^3b^2-a^2b^3+ab^4-b^5)=a^6-b^6.$

Then, if we multiply both terms by

$$(2^2)^{\frac{5}{6}}-(2^2)^{\frac{4}{6}}(5^3)^{\frac{1}{6}}+(2^2)^{\frac{3}{6}}(5^3)^{\frac{2}{6}}-(2^2)^{\frac{2}{6}}(5^3)^{\frac{3}{6}}+(2^2)^{\frac{1}{6}}(5^3)^{\frac{4}{6}}-(5^3)^{\frac{5}{6}},$$

the denominator will become rational; thus,

$$\frac{1}{(2^2)^{\frac{1}{6}}+(5^3)^{\frac{1}{6}}}=\frac{2^{\frac{5}{3}}-2^{\frac{8}{6}}\cdot 5^{\frac{3}{6}}+2\cdot 5-2^{\frac{4}{6}}\cdot 5^{\frac{9}{6}}+2^{\frac{1}{3}}\cdot 5^2-5^{\frac{5}{2}}}{(2^2)^{\frac{6}{6}}+(5^3)^{\frac{6}{6}}}$$

$$=\frac{2\cdot 2^{\frac{2}{3}}-2\cdot 2^{\frac{2}{6}}\cdot 5^{\frac{3}{6}}+10-2^{\frac{4}{6}}\cdot 5\cdot 5^{\frac{3}{6}}+2^{\frac{1}{3}}\cdot 5^2-5^2\cdot 5^{\frac{1}{2}}}{2^2+5^3}$$

$$=\frac{2\sqrt[3]{2^2}-2\sqrt[6]{2^2\cdot 5^3}+10-5\sqrt[6]{2^4\cdot 5^3}+25\sqrt[3]{2}-25\sqrt{5}}{4+125}$$

$$=\frac{10+2\sqrt[3]{4}-2\sqrt[6]{500}-5\sqrt[6]{2000}+25\sqrt[3]{2}-25\sqrt{5}}{129}.$$

EXERCISE 194

Reduce each of the following to an equivalent fraction having a rational denominator:

1. $\dfrac{1}{\sqrt[3]{a}+\sqrt[3]{b}}.$

2. $\dfrac{1}{2-\sqrt[3]{4}}.$

3. $\dfrac{1}{m-\sqrt[4]{n}}.$

4. $\dfrac{1}{\sqrt[4]{3}+\sqrt[4]{4}}.$

5. $\dfrac{1}{\sqrt{a}-\sqrt[4]{b}}.$

6. $\dfrac{\sqrt[3]{3}-\sqrt{2}}{\sqrt[3]{3}+\sqrt{2}}.$

THE BINOMIAL THEOREM FOR POSITIVE INTEGRAL EXPONENTS

447. In the proof of § 387, we only considered the first four terms of the expansion of $(a+x)^{n+1}$, in equation (2).

To make the proof complete, we must show that the fifth law of § 386 holds for any two consecutive terms, in equation (2).

Let P, Q, and R denote the coefficients of the terms involving $a^{n-r}x^r$, $a^{n-r-1}x^{r+1}$, and $a^{n-r-2}x^{r+2}$, respectively, in the second member of (1), § 386.

Thus, $(a+x)^n = a^n + na^{n-1}x + \cdots$

$$+ Pa^{n-r}x^r + Qa^{n-r-1}x^{r+1} + Ra^{n-r-2}x^{r+2} + \cdots. \quad (3)$$

Multiplying both members by $a+x$, we have

$$\begin{aligned}(a+x)^{n+1} = a^{n+1} + na^n x + \cdots + Qa^{n-r}x^{r+1} + Ra^{n-r-1}x^{r+2} + \cdots \\ + a^n x + \cdots + Pa^{n-r}x^{r+1} + Qa^{n-r-1}x^{r+2} + \cdots \\ = a^{n+1} + (n+1)a^n x + \cdots \\ + (P+Q)a^{n-r}x^{r+1} + (Q+R)a^{n-r-1}x^{r+2} + \cdots. \quad (4)\end{aligned}$$

Since the fifth law of § 386 is assumed to hold with respect to the second member of (3), we have

$$Q = \frac{P(n-r)}{r+1}, \text{ and } R = \frac{Q(n-r-1)}{r+2}.$$

Therefore,

$$\frac{Q+R}{P+Q} = \frac{Q + \dfrac{Q(n-r-1)}{r+2}}{\dfrac{Q(r+1)}{n-r} + Q} = \frac{\dfrac{Q(n+1)}{r+2}}{\dfrac{Q(n+1)}{n-r}} = \frac{n-r}{r+2}.$$

Whence, $$Q + R = (P+Q)\frac{n-r}{r+2}.$$

But $n-r$ is the exponent of a in that term of (4) whose coefficient is $P+Q$, and $r+2$ is the exponent of x increased by 1.

Therefore, the fifth law holds with respect to any two consecutive terms in equation (2), § 387.

THE THEOREM OF UNDETERMINED COEFFICIENTS

448. Before giving the more rigorous proof of the Theorem of Undetermined Coefficients, we will prove two theorems in regard to infinite series.

First, if the infinite series

$$a + bx + cx^2 + dx^3 + \cdots$$

is convergent for some finite value of x, it is *finite* for this value of x (§ 393), and therefore finite when $x = 0$.

Hence, the series is convergent when $x = 0$.

449. Second, if the infinite series

$$ax + bx^2 + cx^3 + \cdots$$

is convergent for some finite value of x, it equals 0 when $x = 0$.

For, $ax + bx^2 + cx^3 + \cdots$ is finite for this value of x, and hence $a + bx + cx^2 + \cdots$ is finite for this value of x.

Then, $a + bx + cx^2 + \cdots$ is finite when $x = 0$; and therefore $x(a + bx + cx^2 + \cdots)$, or $ax + bx^2 + cx^3 + \cdots$, equals 0 when $x = 0$.

450. Proof of the Theorem of Undetermined Coefficients (§ 396).

The equation

$$A + Bx + Cx^2 + Dx^3 + \cdots = A' + B'x + C'x^2 + D'x^3 + \cdots \quad (1)$$

is satisfied when x has any value which makes both members convergent; and since both members are convergent when $x = 0$ (§ 448), the equation is satisfied when $x = 0$.

Putting $x = 0$, we have by § 449,

$$Bx + Cx^2 + Dx^3 + \cdots = 0, \text{ and } B'x + C'x^2 + D'x^3 + \cdots = 0.$$

Whence, $\quad A = A'.$

Subtracting A from the first member of (1), and its equal A' from the second member, we have

$$Bx + Cx^2 + Dx^3 + \cdots = B'x + C'x^2 + D'x^3 + \cdots.$$

Dividing each term by x,

$$B + Cx + Dx^2 + \cdots = B' + C'x + D'x^2 + \cdots. \quad (2)$$

The members of this equation are finite for the same values of x as the given series (§ 449).

Then, they are convergent, and therefore *equal*, for the same values of x as the given series.

Then the equation (2) is satisfied when $x = 0$.

Putting $x = 0$, we have $B = B'$.

Proceeding in this way, we may prove $C = C'$, etc.

XXXIII. THE FUNDAMENTAL LAWS FOR ADDITION AND MULTIPLICATION

451. The Commutative Law for Addition.

If a man gains \$ 8, then loses \$ 3, then gains \$ 6, and finally loses \$ 2, the effect on his property will be the same in whatever *order* the transactions occur.

Then, with the notation of § 16, the result of adding + \$ 8, − \$ 3, + \$ 6, and − \$ 2, will be the same in whatever order the transactions occur.

Then, omitting reference to the unit, the result of adding $+8$, -3, $+6$, and -2 will be the same in whatever order the numbers are taken.

This is the Commutative Law for Addition, which is:

The sum of any set of numbers will be the same in whatever order they may be added.

452. The Associative Law for Addition.

The result of adding $b+c$ to a is expressed $a+(b+c)$, which equals $(b+c)+a$ by the Commutative Law for Addition (§ 451).

But $(b+c)+a$ equals $b+c+a$, by the definition of § 3; and $b+c+a$ equals $a+b+c$, by the Commutative Law for Addition.

Whence, $$a+(b+c)=a+b+c.$$

Then, *to add the sum of a set of numbers, we add the numbers separately.*

This is the Associative Law for Addition.

453. The Commutative Law for Multiplication.

The product of a set of numbers will be the same in whatever order they may be multiplied.

By § 55, the *sign* of the product of any number of terms is independent of their order; hence, it is sufficient to prove the commutative law for *arithmetical numbers.*

Let there be, in the figure, a stars in each row, and b rows.

We may find the entire number of stars by multiplying the number in each row, a, by the number of rows, b.

a in a row.
* * * * ...
* * * * ...
* * * * ...
. . . .
b rows.

Thus, the entire number of stars is $a \times b$.

We may also find the entire number of stars by multiplying the number in each vertical column, b, by the number of columns, a.

Thus, the entire number of stars is $b \times a$.

Therefore, $$a \times b = b \times a.$$

This proves the law for the product of two positive integers.

Again, let c, d, e, and f be any positive integers.

Then, $\frac{c}{d} \times \frac{e}{f} = \frac{c \times e}{d \times f}$; for, to multiply two fractions, we multiply the numerators together for the numerator of the product, and the denominators together for its denominator.

Then, $\frac{c}{d} \times \frac{e}{f} = \frac{e \times c}{f \times d}$; since the commutative law for multiplication holds for the product of two positive integers.

Hence, $\frac{c}{d} \times \frac{e}{f} = \frac{e}{f} \times \frac{c}{d}$; which proves the commutative law for the product of two positive fractions.

454. The Associative Law for Multiplication.

To multiply by the product of a set of numbers, we multiply by the numbers of the set separately.

This law was assumed to hold in §§ 56 and 57.

The result of multiplying a by bc is expressed $a \times (bc)$, which equals $(bc) \times a$, by the Commutative Law for Multiplication.

But by the definition of § 5, $(bc) \times a$ equals bca, which equals abc by the Commutative Law for Multiplication.

Whence, $$a \times (bc) = abc.$$

This proves the law for the product of three numbers.

The Commutative and Associative Laws for Multiplication may be **proved for the product of any number of arithmetical numbers.**

(See the author's Advanced Course in Algebra, §§ 18 and 19.)

455. The Distributive Law for Multiplication.

The law is expressed $(a+b)c=ac+bc$ (§ 40).

We will now prove this result for all values of a, b, and c.

I. Let a and b have any values, and let c be a positive integer.

Then, $(a+b)c=(a+b)+(a+b)+\cdots$ to c terms

$$=(a+a+\cdots \text{ to } c \text{ terms})+(b+b+\cdots \text{ to } c \text{ terms})$$

(by the Commutative and Associative Laws for Addition),

$$=ac+bc.$$

II. Let a and b have any values, and let $c=\frac{e}{f}$, where e and f are positive integers.

Since the product of the quotient and divisor equals the dividend,

$$\frac{e}{f}\times f=e.$$

Then, $(a+b)\times\frac{e}{f}\times f=(a+b)\times e=ae+be$, by I.

Whence, $(a+b)\times\frac{e}{f}\times f=a\times\frac{e}{f}\times f+b\times\frac{e}{f}\times f.$

Dividing each term by f (Ax. 8, § 9), we have

$$(a+b)\times\frac{e}{f}=a\times\frac{e}{f}+b\times\frac{e}{f}.$$

Thus, the result is proved when c is a positive integer or a positive fraction.

III. Let a and b have any values, and let $c=-g$, where g is a positive integer or fraction.

By § 54, $(a+b)(-g)=-(a+b)g=-(ag+bg)$, by I and II,

$$=-ag-bg=a(-g)+b(-g).$$

Thus, the distributive law is proved for all positive or negative, integral or fractional, values of a, b, and c.

XXXIV. ADDITIONAL METHODS IN FACTORING

456. The Remainder Theorem.

Let it be required to divide $px^2 + qx + r$ by $x - a$.

$$\begin{array}{l|l} px^2 + \quad qx + r & x - a \\ \cline{2-2} \underline{px^2 - apx} & px + (ap + q) \\ (ap+q)x & \\ \underline{(ap+q)x - pa^2 - qa} & \\ \qquad\qquad pa^2 + qa + r, \text{ Remainder.} & \end{array}$$

We observe that the final remainder,

$$pa^2 + qa + r,$$

is the same as the dividend with a substituted in place of x; this exemplifies the following law:

If any polynomial, involving x, be divided by $x - a$, the remainder of the division equals the result obtained by substituting a for x in the given polynomial.

This is called *The Remainder Theorem.*

To prove the theorem, let

$$px^n + qx^{n-1} + \cdots + rx + s$$

be any polynomial involving x.

Let the division of the polynomial by $x - a$ be carried on until a remainder is obtained which does not contain x.

Let Q denote the quotient, and R the remainder.

Since the dividend equals the product of the quotient and divisor, plus the remainder, we have

$$Q(x-a) + R = px^n + qx^{n-1} + \cdots + rx + s.$$

Putting x equal to a, in the above equation, we have,

$$R = pa^n + qa^{n-1} + \cdots + ra + s.$$

457. The Factor Theorem.

If any polynomial, involving x, becomes zero when x is put equal to a, the polynomial has $x-a$ as a factor.

For, by § 456, if the polynomial is divided by $x-a$, the remainder is zero.

458. Examples.

1. Find whether $x-2$ is a factor of x^3-5x^2+8.

Substituting 2 for x, the expression x^3-5x^2+8 becomes

$$2^3-5\cdot 2^2+8, \text{ or } -4.$$

Then, by § 456, if x^3-5x^2+8 be divided by $x-2$, the remainder is -4; and $x-2$ is not a factor.

2. Find whether $m+n$ is a factor of

$$m^4-4m^3n+2m^2n^2+5mn^3-2n^4. \qquad (1)$$

Putting $m=-n$, the expression becomes

$$n^4+4n^4+2n^4-5n^4-2n^4, \text{ or } 0.$$

Then, by § 456, if the expression (1) be divided by $m+n$, the remainder is 0; and $m+n$ is a factor.

3. Prove that a is a factor of

$$(a+b+c)(ab+bc+ca)-(a+b)(b+c)(c+a).$$

Putting $a=0$, the expression becomes

$$(b+c)bc-b(b+c)c, \text{ or } 0.$$

Then, by § 456, $a-0$, or a, is a factor of the expression.

4. Factor $x^3-3x^2-14x-8$.

The positive and negative integral factors of 8 are 1, 2, 4, 8, -1, -2, -4, and -8.

It is best to try the numbers in their order of absolute magnitude.

If $x=1$, the expression becomes $1-3-14-8$.

If $x=-1$, the expression becomes $-1-3+14-8$.

If $x=2$, the expression becomes $8-12-28-8$.

If $x=-2$, the expression becomes $-8-12+28-8$, or 0.

This shows that $x+2$ is a factor.

Dividing the expression by $x+2$, the quotient is x^2-5x-4.

Then, $$x^3-3x^2-14x-8=(x+2)(x^2-5x-4).$$

EXERCISE 195

Factor the following:

1. x^5+1.
2. x^4-81.
3. x^6-64.
4. $x^3+4x^2+7x-12$.
5. $x^4-x^3+6x^2+14x+6$.
6. $x^3-x^2-11x-10$.
7. $x^3-9x^2+15x+9$.
8. $x^3-18x+8$.
9. $x^3-5x^2-8x+48$.
10. $x^4+8x^3+13x^2-13x-4$.
11. x^3+6x^2-x-30.

Find, without actual division,

12. Whether $x-3$ is a factor of $x^3-6x^2+13x-12$.
13. Whether $x+2$ is a factor of x^3+7x^2-6.
14. Whether $x+1$ is a factor of $x^4-4x^3+2x^2-2x-9$.
15. Whether x is a factor of $x(y+z)^2+y(z+x)^2+z(x+y)^2$.
16. Whether a is a factor of $a^3(b-c)^3+b^3(c-a)^3+c^3(a-b)^3$.
17. Whether $x-y$ is a factor of $(x-y)^3+(y-z)^3+(z-x)^3$.
18. Whether $m+n$ is a factor of $m(m+2n)^3-n(2m+n)^3$.

459. We will now give formal proofs of the statements of § 104.

Proof of I.

If b be substituted for a in a^n-b^n, the result is b^n-b^n, or 0.
Then, by § 457, a^n-b^n has $a-b$ as a factor.

Proof of II.

If $-b$ be substituted for a in a^n-b^n, the result is $(-b)^n-b^n$; or, since n is even, b^n-b^n, or 0.

Then, by § 457, a^n-b^n has $a+b$ as a factor.

Proof of III.

If $-b$ be substituted for a in a^n+b^n, the result is $(-b)^n+b^n$; or, since n is odd, $-b^n+b^n$, or 0.

Then, a^n+b^n has $a+b$ as a factor.

Proof of IV.

If $-b$ or $+b$ be substituted for a in a^n+b^n, the results are $(-b)^n+b^n$ or b^n+b^n, respectively.

Since n is even, neither of these is zero.

Then, neither $a+b$ nor $a-b$ is a factor of a^n+b^n.

SYMMETRY

460. An expression containing two or more letters is said to be *symmetrical* with respect to them, when any two of them can be interchanged without altering the value of the expression.

Thus, $ab+bc+ca$ is symmetrical with respect to the letters a, b, and c; for if a and b be interchanged, the expression becomes $ba+ac+cb$, which is equal to $ab+bc+ca$.

And, in like manner, the expression is not altered in value if we interchange b and c, or c and a.

461. Cyclo-symmetry.

An expression containing n letters $a, b, c, \cdots, m, n$, is said to be *cyclo-symmetrical* with respect to them when, if a be replaced by b, b by c, $\cdots$, m by n, and n by a, the value of the expression is not changed.

The above is called a *cyclical* interchange of letters.

Thus, the expression $a^2b+b^2c+c^2a$ is cyclo-symmetrical with respect to the letters a, b, and c; for if a be replaced by b, b by c, and c by a, the expression becomes $b^2c+c^2a+a^2b$, which is equal to $a^2b+b^2c+c^2a$.

The above expression is not symmetrical with respect to a, b, and c; for if a and b be interchanged, the expression becomes $b^2a+a^2c+c^2b$, which is not equal to $a^2b+b^2c+c^2a$.

462. It follows from §§ 460 and 461 that, if two expressions are symmetrical or cyclo-symmetrical, the results obtained by adding, subtracting, multiplying, or dividing them are, respectively, symmetrical or cyclo-symmetrical.

463. Applications.

The principle of symmetry is often useful in abridging algebraic operations.

1. Expand $(a+b+c)^3$.

We have, $(a+b+c)^3 = (a+b+c)(a+b+c)(a+b+c)$.

This expression is symmetrical with respect to a, b, and c (§ 460), and of the third degree.

There are three possible types of terms of the third degree in a, b, c; terms like a^3, terms like a^2b, and terms like abc.

It is evident that a^3 has the coefficient 1; and so, by symmetry, b^3 and c^3 have the coefficient 1.

The a^2b terms may be obtained by multiplying the a's in any two factors by the b in the remaining factor.

Then, it is evident that a^2b has the coefficient 3; and so, by symmetry, have b^2a, b^2c, c^2b, c^2a, and a^2c.

Let m denote the coefficient of abc.

Then, $(a+b+c)^3$

$$= a^3 + b^3 + c^3 + 3(a^2b + b^2a + b^2c + c^2b + c^2a + a^2c) + mabc.$$

To determine m, we observe that the above equation holds for all values of a, b, and c.

We may therefore let $a = b = c = 1$.

Then, $27 = 3 + 18 + m$; and $m = 6$.

Whence, $(a+b+c)^3$

$$= a^3 + b^3 + c^3 + 3(a^2b + b^2a + b^2c + c^2b + c^2a + a^2c) + 6\,abc.$$

2. Expand $(x-y-z)^2 + (y-z-x)^2 + (z-x-y)^2$.

This expression is symmetrical with respect to x, y, and z, and of the second degree.

The possible types of terms of the second degree in x, y, and z are terms like x^2, and terms like xy.

It is evident, by the rule of § 204, that x^2 has the coefficient 3; and so, by symmetry, have y^2 and z^2.

Let m denote the coefficient of xy.

Then, $(x-y-z)^2 + (y-z-x)^2 + (z-x-y)^2$

$$= 3(x^2 + y^2 + z^2) + m(xy + yz + zx).$$

To determine m, put $x = y = z = 1$.

Then, $3 = 9 + 3m$, or $m = -2$.

Whence, $(x-y-z)^2 + (y-z-x)^2 + (z-x-y)^2$

$$= 3(x^2 + y^2 + z^2) - 2(xy + yz + zx).$$

3. Expand

$$(a+b+c)^3+(a+b-c)^3+(b+c-a)^3+(c+a-b)^3.$$

The expression is symmetrical with respect to a, b, and c, and of the third degree.

The possible types of terms are terms like a^3, terms like a^2b, and terms like abc.

It is evident, by proceeding as in Ex. 1, that a^3 has the coefficient $1+1-1+1$, or 2; and so, by symmetry, have b^3 and c^3.

Again, proceeding as in Ex. 1, it is evident that a^2b has the coefficient 3 in the first term, 3 in the second, 3 in the third, and -3 in the fourth.

Then, a^2b has the coefficient $3+3+3-3$, or 6; and so by symmetry have b^2a, b^2c, c^2b, c^2a, and a^2c.

Let m denote the coefficient of abc.

Then, $(a+b+c)^3+(a+b-c)^3+(b+c-a)^3+(c+a-b)^3$

$$=2(a^3+b^3+c^3)+6(a^2b+b^2a+b^2c+c^2b+c^2a+a^2c)+mabc.$$

To determine m, let $a=b=c=1$.

Then, $27+1+1+1=6+36+m$, or $m=-12$.

Then, $(a+b+c)^3+(a+b-c)^3+(b+c-a)^3+(c+a-b)^3$

$$=2(a^3+b^3+c^3)+6(a^2b+b^2a+b^2c+c^2b+c^2a+a^2c)-12\,abc.$$

EXERCISE 196

1. In the expansion of an expression which is symmetrical with respect to a, b, and c, what are the possible types of terms of the fourth degree? of the fifth degree?

2. If one term of an expression which is symmetrical with respect to a, b, and c is $(2a-b-c)(2b-c-a)$, what are the others?

3. Is the expression $a(b-c)^2+b(c-a)^2+c(a-b)^2$ symmetrical with respect to a, b, and c?

4. Is the expression $(x^2-y^2)^3+(y^2-z^2)^3+(z^2-x^2)^3$ symmetrical with respect to x, y, and z?

Expand the following by the symmetrical method:

5. $(a+b+c)^2$. **6.** $(a+b+c+d)^2$.

7. $(x+y-z)^2+(y+z-x)^2+(z+x-y)^2$.

8. $(2a-3b-4c)^2+(2b-3c-4a)^2+(2c-3a-4b)^2$.

9. $(a+b+c)^3+(a-b-c)^3+(b-c-a)^3+(c-a-b)^3$.

10. $(a+b+c-d)^2+(b+c+d-a)^2+(c+d+a-b)^2$
$+(d+a+b-c)^2$.

11. $(a+b+c+d)^3$.

12. $(x+y-z)(y+z-x)(z+x-y)$.

13. $(a+b+c)(a+b-c)(b+c-a)(c+a-b)$.

14. $(x^2+y^2+z^2+2xy+2yz+2zx)^2$.

464. Factoring of Symmetrical Expressions.

The method of § 457 is advantageous in factoring symmetrical expressions (§§ 460, 461).

1. Factor

$$a(b+c)^2+b(c+a)^2+c(a+b)^2-a^2(b+c)-b^2(c+a)-c^2(a+b).$$

The expression is symmetrical with respect to a, b, and c.

Being of the third degree, the only literal factors which it can have are three of the type a; three of the type $a+b$; or $a+b+c$, and a factor of the second degree.

Putting $a=0$, the expression becomes

$$bc^2+cb^2-b^2c-c^2b, \text{ or } 0.$$

Then, by § 457, a is a factor; and, by symmetry, b and c are factors.

The expression, being of the third degree, can have no other literal factor; but it may have a *numerical* factor.

Let the given expression $= mabc$.

To determine m, let $a=b=c=1$.

Then, $\qquad 4+4+4-2-2-2=m$, or $m=6$.

Whence, the given expression $=6abc$.

2. Factor $x^3+y^3+z^3-3xyz$.

The expression is symmetrical with respect to x, y, and z.

The only literal factors which it can have are three of the type x; three of the type $x+y$; or $x+y+z$, and a factor of the second degree.

It is evident that neither x, y, nor z is a factor.

Putting x equal to $-y$, the expression becomes

$$-y^3 + y^3 + z^3 + 3x^2z,$$

which is not 0.

Then, $x + y$ is not a factor (§ 457); and, by symmetry, neither $y + z$ nor $z + x$ is a factor.

Putting x equal to $-y - z$, the expression becomes

$$(-y-z)^3 + y^3 + z^3 - 3(-y-z)yz$$
$$= -y^3 - 3y^2z - 3yz^2 - z^3 + y^3 + z^3 + 3y^2z + 3yz^2 = 0.$$

Therefore, $x + y + z$ is a factor.

The other factor may be obtained by division, or by the following process:

It is of the second degree; and as it is symmetrical with respect to x, y, and z, it must be of the form

$$m(x^2 + y^2 + z^2) + n(xy + yz + zx).$$

It is evident that $m = 1$, as this is the only value which will give the terms x^3, y^3, and z^3 in the given expression.

Then,

$$x^3 + y^3 + z^3 - 3xyz = (x+y+z)[x^2 + y^2 + z^2 + n(xy + yz + zx)].$$

To determine n, let $x = 1$, $y = 1$, $z = 0$.

Then, $2 = 2(2 + n)$, or $1 = 2 + n$, or $n = -1$.

Whence,

$$x^3 + y^3 + z^3 - 3xyz = (x+y+z)(x^2 + y^2 + z^2 - xy - yz - zx).$$

3. Factor $ab(a-b) + bc(b-c) + ca(c-a)$.

The expression is cyclo-symmetrical (§ 461) with respect to a, b, and c.

It is evident that neither a, b, nor c is a factor.

The expression becomes 0 when a is replaced by b.

Then, $a - b$ is a factor; and, by symmetry, $b - c$ and $c - a$ are factors.

The expression can have no other literal factor, but may have a numerical one.

Let the given expression $= m(a-b)(b-c)(c-a)$.

To determine m, let $a = 2$, $b = 1$, and $c = 0$.

Then, $2 = -2m$, and $m = -1$.

Then, the given expression $= -(a-b)(b-c)(c-a)$.

EXERCISE 197

Factor the following:

1. $m^3 + 2\,m^2n + 2\,mn^2 + n^3.$
2. $(ab+bc+ca)(a+b+c)-a^2(b+c)-b^2(c+a)-c^2(a+b).$
3. $x^2(y+z)+y^2(z+x)+z^2(x+y)+2\,xyz.$
4. $a(b+c)^2+b(c+a)^2+c(a+b)^2-4\,abc.$
5. $a^2(b-c)+b^2(c-a)+c^2(a-b).$
6. $(x+y+z)(xy+yz+zx)-(x+y)(y+z)(z+x)$
7. $ab(a+b)+bc(b+c)+ca(c+a)+2\,abc.$
8. $(x+y+z)^3-x^3-y^3-z^3.$
9. $(x+y+z)(xy+yz+zx)-xyz.$
10. $(x-y)^3+(y-z)^3+(z-x)^3.$
11. $a^3(b-c)+b^3(c-a)+c^3(a-b).$

XXXV. MATHEMATICAL INDUCTION

465. In § 387 we gave an example of Mathematical Induction, in proving the Binomial Theorem for a Positive Integral Exponent; in the present chapter, we will give other illustrations of the method.

466. We will now prove that the laws of § 103 hold universally.

We will first prove, by Mathematical Induction, that they hold for $\frac{a^n - b^n}{a - b}$, where n is any positive integer.

Assume the laws to hold for $\frac{a^n - b^n}{a - b}$, where n is any positive integer.

Then,
$$\frac{a^n - b^n}{a - b} = a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \cdots + b^{n-1}. \quad (1)$$

Now,
$$\frac{a^{n+1} - b^{n+1}}{a - b} = \frac{a^{n+1} - a^n b + a^n b - b^{n+1}}{a - b}$$
$$= \frac{a^n(a - b) + b(a^n - b^n)}{a - b}$$
$$= a^n + b(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \cdots + b^{n-1}), \text{ by (1)},$$
$$= a^n + a^{n-1}b + a^{n-2}b^2 + \cdots + b^n.$$

This result is in accordance with the laws of § 103.

Hence, if the laws hold for the quotient of the difference of two like powers of a and b divided by $a - b$, they also hold for the quotient of the difference of the next higher powers of a and b divided by $a - b$.

But we know that they hold for $\frac{a^5 - b^5}{a - b}$, and therefore they hold for $\frac{a^6 - b^6}{a - b}$; and since they hold for $\frac{a^6 - b^6}{a - b}$, they hold for $\frac{a^7 - b^7}{a - b}$; and so on.

Hence, the laws hold for $\frac{a^n - b^n}{a - b}$, where n is any positive integer.

Putting $-b$ for b in (1), we have

$$\frac{a^n - (-b)^n}{a - (-b)} = a^{n-1} + a^{n-2}(-b) + \cdots + (-b)^{n-1}.$$

If n is even, $(-b)^n = b^n$, and $(-b)^{n-1} = -b^{n-1}$.

Whence, $$\frac{a^n - b^n}{a + b} = a^{n-1} - a^{n-2}b + a^{n-3}b^2 - \cdots - b^{n-1}. \qquad (2)$$

If n is odd, $(-b)^n = -b^n$, and $(-b)^{n-1} = +b^{n-1}$.

Whence, $$\frac{a^n + b^n}{a + b} = a^{n-1} - a^{n-2}b + a^{n-3}b^2 - \cdots + b^{n-1}. \qquad (3)$$

Equations (2) and (3) are in accordance with the laws of § 103.

467. We will now prove that the law of § 204 holds for the square of a polynomial of any number of terms.

Assume the law to hold for the square of a polynomial of m terms, where m is any positive integer; that is,

$$\begin{aligned} &(a + b + c + \cdots + l + m)^2 \\ &= a^2 + b^2 + \cdots + m^2 + 2a(b + c + \cdots + m) \\ &\quad + 2b(c + \cdots + m) + \cdots + 2lm. \end{aligned} \qquad (1)$$

Then, $$\begin{aligned} &(a + b + c + \cdots + m + n)^2 \\ &= [(a + b + c + \cdots + m) + n]^2 \\ &= (a + b + c + \cdots + m)^2 \\ &\quad + 2(a + b + c + \cdots + m)n + n^2, \text{ by § 97,} \\ &= a^2 + b^2 + c^2 + \cdots + m^2 + n^2 \\ &\quad + 2a(b + c + \cdots + m + n) \\ &\quad + 2b(c + \cdots + m + n) + \cdots + 2mn, \text{ by (1).} \end{aligned}$$

This result is in accordance with the law of § 204.

Hence, if the law holds for the square of a polynomial of m terms, where m is any positive integer, it also holds for the square of a polynomial of $m+1$ terms.

But we know that the law holds for the square of a polynomial of three terms, and therefore it holds for the square of a polynomial of four terms; and since it holds for the square of a polynomial of four terms, it also holds for the square of a polynomial of five terms; and so on.

Hence, the law holds for the square of any polynomial.

468. As another illustration of the method, we will prove that the sum of the first n terms of the arithmetic progression,

$$a, \ a+d, \ a+2d, \ \cdots,$$

is given by the formula $na+\frac{n(n-1)}{2}d$. (Compare § 361.)

The sum of the first two terms is $2a+d$, which can be written in the form $2a+\frac{2(2-1)}{2}d$.

Then, the formula holds for the sum of the first two terms.

Assume that the formula holds for the sum of the first n terms.

That is, the sum of the first n terms $=na+\frac{n(n-1)}{2}d$.

Now the $(n+1)$th term of the progression is $a+nd$.

Whence, the sum of the first $(n+1)$ terms equals

$$na+\frac{n(n-1)}{2}d+a+nd=(n+1)a+\frac{nd}{2}(n-1+2)$$

$$=(n+1)a+\frac{(n+1)n}{2}d.$$

This result is in accordance with the formula.

Hence, if the formula holds for the sum of the first n terms, it also holds for the sum of the first $n+1$ terms.

But we know that the formula holds for the sum of the first two terms, and hence it holds for the sum of the first three terms; and since it holds for the sum of the first three terms, it also holds for the sum of the first four terms; and so on.

Hence, the formula holds for the sum of the first n terms, where n is any positive integer.

EXERCISE 198

1. Prove that the sum of the first n terms of the series 1, 3, 5, $\cdots$ is n^2.

2. Prove that the sum of the first n terms of the series 3, 6, 9, $\cdots$ is $\frac{3n(n+1)}{2}$.

3. Prove that the sum of the first n terms of the series $\frac{1}{1\cdot 2}, \frac{1}{2\cdot 3}, \frac{1}{3\cdot 4}, \cdots$ is $\frac{n}{n+1}$.

4. Prove, by mathematical induction, that the sum of the first n terms of the geometric progression,

$$a,\ ar,\ ar^2,\ \cdots,$$

is given by the formula $S = \frac{a(r^n - 1)}{r - 1}$ (§ 370).

5. Prove that the sum of the first n terms of the series 2^2 4^2, 6^2, $\cdots$ is $\frac{2n(n+1)(2n+1)}{3}$.

6. Prove that the sum of the first n terms of the series 1^3, 2^3, 3^3, $\cdots$ is $\frac{n^2(n+1)^2}{4}$.

XXXVI. EQUIVALENT EQUATIONS

469. Two equations, each involving one or more unknown numbers, are said to be **Equivalent** when every solution of the first is a solution of the second, and every solution of the second a solution of the first.

470. To solve an equation involving one unknown number, x, we transform it into a series of equations, which lead finally to the value of x.

We have assumed, in passing from any equation to any other, in this series, that every solution of the first was a solution of the second, and every solution of the second a solution of the first; so that it was legitimate to use the second in place of the first to find the value of the unknown number.

That is, we have assumed that the two equations were equivalent (§ 469).

We will now prove some theorems in regard to equivalent equations.

471. *If the same expression be added to both members of an equation, the resulting equation will be equivalent to the first.*

Let $$A = B \qquad (1)$$

be an equation involving one or more unknown numbers.

To prove the equation $A + C = B + C$, (2)

where C is any expression, equivalent to (1).

Any solution of (1), when substituted for the unknown numbers, makes A identically equal to B (§ 79).

It then makes $A + C$ identically equal to $B + C$ (§ 84, 1).

Then it is a solution of (2).

Again, any solution of (2), when substituted for the unknown numbers, makes $A + C$ identically equal to $B + C$.

It then makes A identically equal to B (§ 84, 2).

Then it is a solution of (1).

Therefore, (1) and (2) are equivalent.

The principle of § 84, 1, is a special case of the above.

472. The demonstration of § 471 also proves that

If the same expression be subtracted from both members of an equation, the resulting equation will be equivalent to the first.

The principle of § 84, 2, is a special case of this.

473. *If the members of an equation be multiplied by the same expression, which is not zero, and does not involve the unknown numbers, the resulting equation will be equivalent to the first.*

Let $$A = B \qquad (1)$$

be an equation involving one or more unknown numbers.

To prove the equation $A \times C = B \times C$, (2)

where C is not zero, and does not involve the unknown numbers, equivalent to (1).

Any solution of (1), when substituted for the unknown numbers, makes A identically equal to B.

It then makes $A \times C$ identically equal to $B \times C$ (§ 84, 3).

Then it is a solution of (2).

Again, any solution of (2), when substituted for the unknown numbers, makes $A \times C$ identically equal to $B \times C$.

It then makes A identically equal to B (§ 84, 4).

Then it is a solution of (1).

Therefore, (1) and (2) are equivalent.

The reason why the above does not hold for the multiplier zero is, that the principle of § 84, 4, does not hold when the divisor is zero.

The principle of § 84, 3, is a special case of the above.

474. If the members of an equation be multiplied by an expression which involves the unknown numbers, the resulting equation is, in general, not equivalent to the first.

Consider, for example, the equation $x + 2 = 3x - 4$. (1)

Now the equation

$$(x+2)(x-1)=(3x-4)(x-1), \tag{2}$$

which is obtained from (1) by multiplying both members by $x-1$, is satisfied by the value $x=1$, which does not satisfy (1).

Then (1) and (2) are not equivalent.

Thus it is never allowable to multiply both members of an integral equation by an expression which involves the unknown numbers; for in this way additional solutions are introduced.

475. *If the members of an equation be divided by the same expression, which is not zero, and does not involve the unknown numbers, the resulting equation will be equivalent to the first.*

Let
$$A=B \tag{1}$$
be an equation involving one or more unknown numbers.

To prove the equation
$$\frac{A}{C}=\frac{B}{C}, \tag{2}$$
where C is not zero, and does not involve the unknown numbers, equivalent to (1).

Any solution of (1), when substituted for the unknown numbers, makes A identically equal to B.

It then makes $\frac{A}{C}$ identically equal to $\frac{B}{C}$ (§ 84, 4).

Then it is a solution of (2).

Again, any solution of (2), when substituted for the unknown numbers, makes $\frac{A}{C}$ identically equal to $\frac{B}{C}$.

It then makes A identically equal to B.

Then it is a solution of (1).

Therefore, (1) and (2) are equivalent.

The principle of § 84, 4, is a special case of the above.

476. If the members of an equation be divided by an expression which involves the unknown numbers, the resulting equation is, in general, not equivalent to the first.

Consider, for example, the equation

$$(x+2)(x-1)=(3x-4)(x-1). \tag{1}$$

Also the equation $$x+2=3x-4, \tag{2}$$

which is obtained from (1) by dividing both members by $x-1$.

Now equation (1) is satisfied by the value $x=1$, which does not satisfy (2).

Then (1) and (2) are not equivalent.

It follows from this that it is never allowable to divide both members of an integral equation by an expression which involves the unknown numbers; for in this way solutions are lost. (Compare § 158.)

477. *If both members of a fractional equation be multiplied by the* L.C.M. *of the given denominators, the resulting equation is, in general, equivalent to the first.*

Let all the terms be transposed to the first member, and let them be added, using for a common denominator the L. C. M. of the given denominators.

The equation will then be in the form

$$\frac{A}{B}=0. \tag{1}$$

We will now prove the equation

$$A=0, \tag{2}$$

which is obtained by multiplying (1) by the L. C. M. of the given denominators, equivalent to (1), *if A and B have no common factor.*

Any solution of (1), when substituted for the unknown numbers, makes $\frac{A}{B}$ identically equal to 0.

Then, it must make A identically equal to 0.

Then, it is a solution of (2).

Again, any solution of (2), when substituted for the unknown numbers, makes A identically equal to 0.

Since A and B have no common factor, B cannot be 0 when this solution is substituted for the unknown numbers.

Then, any solution of (2), when substituted for the unknown numbers, makes $\frac{A}{B}$ identically equal to 0, and is a solution of (1).

Therefore, (1) and (2) are equivalent, if A and B have no common factor.

If A and B have a common factor, (1) and (2) are not equivalent; consider, for example, the equations

$$\frac{x-1}{x^2-1}=0, \text{ and } x-1=0.$$

The second equation is satisfied by the value $x=1$, which does not satisfy the first equation; then, the equations are not equivalent.

478. A fractional equation may be cleared of fractions by multiplying both members by *any* common multiple of the denominators; but in this way additional solutions are introduced, and the resulting equation is not equivalent to the first.

Consider, for example, the equation

$$\frac{x^2}{x^2-1}+\frac{x}{x-1}=2.$$

If we solve by multiplying both members by x^2-1, the L. C. M. of x^2-1 and $x-1$, we find $x=-2$.

If, however, we multiply both members by $(x^2-1)(x-1)$, we have

$$x^3-x^2+x^3-x=2x^3-2x^2-2x+2, \text{ or } x^2+x-2=0.$$

The latter equation may be solved as in § 126.

The factors of x^2+x-2 are $x+2$ and $x-1$.

Solving the equation $x+2=0$, $x=-2$.

Solving the equation $x-1=0$, $x=1$.

This gives the additional value $x=1$; and it is evident that this does not satisfy the given equation.

479. *If both members of an equation be raised to the same positive integral power, the resulting equation will have all the solutions of the given equation, and, in general, additional ones.*

Consider, for example, the equation $x = 3$.

Squaring both members, we have

$$x^2 = 9, \text{ or } x^2 - 9 = 0, \text{ or } (x+3)(x-3) = 0.$$

The latter equation has the root 3, and, in addition, the root -3.

We will now consider the general case.

Let $$A = B \tag{1}$$

be an equation involving one or more unknown numbers.

Raising both members to the nth power, n being a positive integer, we have

$$A^n = B^n, \text{ or } A^n - B^n = 0. \tag{2}$$

Factoring the first number (§ 121),

$$(A-B)(A^{n-1} + A^{n-2}B + \cdots + B^{n-1}) = 0. \tag{3}$$

Now, equation (3) is satisfied when $A = B$.

Whence, equation (2) has all the solutions of (1).

But (3) is also satisfied when

$$A^{n-1} + A^{n-2}B + \cdots + B^{n-1} = 0;$$

so that (2) has also the solutions of this last equation, which, in general, do not satisfy (1).

EQUIVALENT SYSTEMS OF EQUATIONS

480. Two systems of equations, involving two or more unknown numbers, are said to be *equivalent* when every solution of the first system is a solution of the second, and every solution of the second a solution of the first.

481. *If*

$$\begin{cases} A = 0, \\ B = 0, \end{cases}$$

are equations involving two or more unknown numbers, the system of equations

$$\begin{cases} A = 0, \\ mA + nB = 0, \end{cases}$$

where m and n are any numbers, and n not equal to zero, is equivalent to the first system.

For any solution of the first system, when substituted for the unknown numbers, makes $A=0$ and $B=0$.

It then makes $A=0$ and $mA+nB=0$.

Then, it is a solution of the second system.

Again, any solution of the second system, when substituted for the unknown numbers, makes $A=0$ and $mA+nB=0$.

It therefore makes $nB=0$, or $B=0$.

Since it makes $A=0$ and $B=0$, it is a solution of the first system.

Hence, the systems are equivalent.

A similar result holds for a system of any number of equations.

Either m or n may be *negative*.

482. *If either equation, in a system of two, be solved for one of the unknown numbers, and the value found be substituted for this unknown number in the other equation, the resulting system will be equivalent to the first.*

Let
$$\begin{cases} A=B, & (1) \\ C=D, & (2) \end{cases}$$

be equations involving two unknown numbers, x and y.

Let E be the value of x obtained by solving (1).

Let $F=G$ be the equation obtained by substituting E for x in (2).

To prove the system of equations

$$\begin{cases} x=E, & (3) \\ F=G, & (4) \end{cases}$$

equivalent to the first system.

Any solution of the first system satisfies (3), for (3) is only a form of (1).

Also, the values of x and y which form the solution make x and E equal; and hence satisfy the equation obtained by putting E for x in (2).

Then, any solution of the first system satisfies (4).

Again, any solution of the second system satisfies (1), for (1) is only a form of (3).

Also, the values of x and y which form the solution make x and E equal; and hence satisfy the equation obtained by putting x for E in (4).

Then, any solution of the second system satisfies (2).

Hence, the systems are equivalent.

A similar result holds for a system of any number of equations, involving any number of unknown numbers.

483. We will now apply the principles of §§ 481 and 482 to show that the solutions of Ex. 1, § 168, and the examples of §§ 169 and 170 are equivalent to the given equations.

Ex. 1, § 168.

By § 481, the given system is equivalent to the system (1) and (5), or to the system (1) and (6).

By § 482, the system (1) and (6) is equivalent to the system (6) and (7), which is equivalent to the system (6) and (8).

Then, the given system is equivalent to the system (6) and (8).

Ex., § 169.

By § 482, the given system is equivalent to the system (3) and (4), or to the system (3) and (5).

By § 482, the system (3) and (5) is equivalent to (5) and (6).

Ex., § 170.

The given system is equivalent to (3) and (4).

Now any values of x and y which satisfy (3) and (4) also satisfy (3) and (5).

Then, the given system is equivalent to the system (3) and (5), or to (3) and (6).

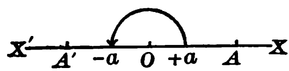
By § 482, the system (3) and (6) is equivalent to (6) and (7).

484. The principles of §§ 471, 472, 473, 475, 477, 479, 480, and 481 hold for equations of any degree.

XXXVII. GRAPHICAL REPRESENTATION OF IMAGINARY NUMBERS

485. Let O be any point in the straight line XX'.

We may suppose any positive real number, $+a$, to be represented by the distance from O to A, a units to the right of O in OX.



Then, with the notation of § 16, any negative real number, $-a$, may be represented by the distance from O to A', a units to the left of O in OX'.

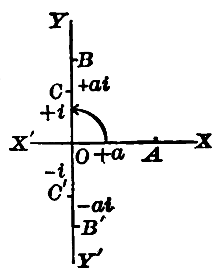
486. Since $-a$ is the same as $(+a)\times(-1)$, it follows from § 485 that the product of $+a$ by -1 is represented by turning the line OA which represents the number $+a$, through two right angles, in a direction opposite to the motion of the hands of a clock.

Then, in the product of any real number by -1, we may regard -1 as an operator which turns the line which represents the first factor through two right angles, in a direction opposite to the motion of the hands of a clock.

487. Graphical Representation of the Imaginary Unit i (§ 276).

By the definition of § 275, $-1=i\times i$.

Then, since one multiplication by i, followed by another multiplication by i, turns the line which represents the first factor through *two* right angles, in a direction opposite to the hands of a clock, we may regard multiplication by i as turning the line through *one* right angle, in the same direction.



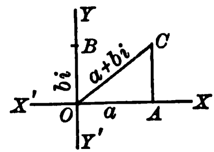
Thus, let XX' and YY' be straight lines intersecting at right angles at O.

Then, if $+a$ be represented by the line OA, where A is a units to the right of O in OX, $+ai$ may be represented by OB, and $-ai$ by OB', where B is a units above, and B' a units below, O, in YY'.

Also, $+i$ may be represented by OC, and $-i$ by OC', where C is one unit above, and C' one unit below, O, in YY'.

488. Graphical Representation of Complex Numbers.

We will now show how to represent the complex number $a+bi$.



Let XX' and YY' be straight lines intersecting at right angles at O.

Let a be represented by OA, to the right of O, if a is positive, to the left if a is negative.

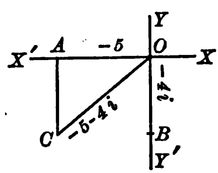
Let bi be represented by OB, above O if b is positive, below if b is negative.

Draw line AC equal and parallel to OB, on the same side of XX' as OB, and line OC.

Then, OC is considered as representing the result of adding bi to a; that is, OC represents the complex number $a+bi$.

The figure represents the case in which both a and b are *positive*.

As another illustration, we will show how to represent the complex number $-5-4i$.



Lay off OA 5 units to the left of O in OX', and OB 4 units below O in YY'.

Draw line AC below XX', equal and parallel to OB, and line OC.

Then, OC represents $-5-4i$.

The complex number $a+bi$, if a is positive and b negative, will be represented by a line between OX and OY'; and if a is negative and b positive, by a line between OY and OX'.

EXERCISE 199

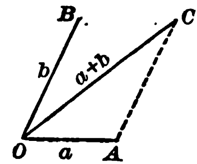
Represent the following graphically:

1. $3i$. **2.** $-6i$. **3.** $4+i$. **4.** $-1+2i$.

5. $2-5i$. **6.** $-5-3i$. **7.** $-7+4i$.

489. Graphical Representation of Addition.

We will now show how to represent the result of adding b to a, where a and b are any two real, pure imaginary, or complex numbers.



Let the line a be represented by OA, and the line b by OB.

Draw the line AC equal and parallel to OB, on the same side of OA as OB, and the line OC.

Then, OC is considered as representing the result of adding b to a; that is, OC represents $a+b$.

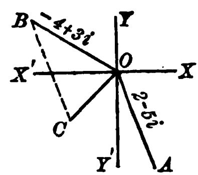
The method of § 488 is a special case of the above.

If a and b are both real, B will fall in OA, or in AO produced through O.

The same will be true if a and b are both pure imaginary.

If one of the numbers, a and b, is real, and the other pure imaginary, the lines OA and OB will be perpendicular.

As another illustration, we will show how to represent graphically the sum of the complex numbers $2-5i$ and $-4+3i$.



The complex number $2-5i$ is represented by the line OA, between OX and OY'.

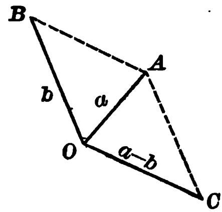
The complex number $-4+3i$ is represented by the line OB, between OY and OX'.

Draw the line BC equal and parallel to OA, on the same side of OB as OA, and the line OC.

Then, the line OC represents the result of adding $-4+3i$ to $2-5i$.

490. Graphical Representation of Subtraction.

Let a and b be any two real, pure imaginary, or complex numbers.



Let a be represented by OA, and b by OB; and complete the parallelogram $OBAC$.

By § 489, OA represents the result of adding the number represented by OB to the number represented by OC.

That is, if b be added to the number represented by OC, the sum is equal to a; hence, $a-b$ is represented by the line OC.

EXERCISE 200

Represent the following graphically:

1. The sum of $4i$ and $3-5i$.

2. The sum of $-5i$ and $-1+6i$.

3. The sum of $2+4i$ and $5-3i$.

4. The sum of $-6+2i$ and $-4-7i$.

5. Represent graphically the result of *subtracting* the second expression from the first, in each of the above examples.

XXXVIII. INDETERMINATE FORMS

491. In § 322, we found that the form $\frac{0}{0}$ indicated an expression which could have *any value whatever;* but this is not always the case.

Consider, for example, the fraction $\frac{x^2-a^2}{x^2-ax}$.

If $x=a$, the fraction takes the form $\frac{0}{0}$.

Now, $$\frac{x^2-a^2}{x^2-ax}=\frac{(x+a)(x-a)}{x(x-a)}=\frac{x+a}{x};$$

which last expression is equal to the given fraction, provided x does not equal a.

The fraction $\frac{x+a}{x}$ approaches the limit $\frac{a+a}{a}$, or 2, when x approaches the limit a.

This limit we call *the value of the given fraction when* $x=a$.

Then, the value of the given fraction when $x=a$ is 2.

In any similar case, we cancel the factor which equals 0 for the given value of x, and find the limit approached by the result when x approaches the given value as a limit.

EXERCISE 201

Find the values of the following:

1. $\dfrac{2ax-4a^2}{x^2-4a^2}$ when $x=2a$.

2. $\dfrac{2x^3-5x^2}{4x^2+3x}$ when $x=0$.

3. $\dfrac{x^2-16}{x^2+2x-8}$ when $x=-4$.

4. $\dfrac{4x^2-4x-3}{6x^2-17x+12}$ when $x=\frac{3}{2}$.

5. $\dfrac{x^3+6x^2+12x+8}{x^4-8x^2+16}$ when $x=-2$.

6. $\dfrac{x^3-3x^2+3x-2}{x^3-7x+6}$ when $x=2$.

492. Other Indeterminate Forms.

Expressions taking the forms $\frac{\infty}{\infty}$, $0 \times \infty$, or $\infty - \infty$, for certain values of the letters involved, are also indeterminate.

1. Find the value of $(x^3+8)\left(1+\frac{1}{x+2}\right)$ when $x=-2$.

This expression takes the form $0 \times \infty$, when $x = -2$ (§ 319).

Now, $(x^3+8)\left(1+\frac{1}{x+2}\right) = x^3+8+\frac{x^3+8}{x+2}$

$$= x^3+8+x^2-2x+4 = x^3+x^2-2x+12.$$

The latter expression approaches the limit $-8+4+4+12$, or 12, when x approaches the limit -2.

This limit we call *the value of the expression when* $x = -2$; then, the value of the expression when $x = -2$, is 12.

In any similar case, we simplify as much as possible before finding the limit.

2. Find the value of $\frac{1}{1-x} - \frac{2x}{1-x^2}$ when $x = 1$.

The expression takes the form $\infty - \infty$, when $x = 1$ (§ 319).

Now, $$\frac{1}{1-x} - \frac{2x}{1-x^2} = \frac{1+x-2x}{1-x^2} = \frac{1-x}{1-x^2} = \frac{1}{1+x}.$$

The latter expression approaches the limit $\frac{1}{2}$ when x approaches the limit 1.

Then, the value of the expression when $x = 1$, is $\frac{1}{2}$.

493. Another example in which the result is indeterminate is the following:

Ex. Find the limit approached by the fraction $\frac{1+2x}{2-5x}$ when x is indefinitely increased.

Both numerator and denominator increase indefinitely in absolute value when x is indefinitely increased.

Dividing each term of the fraction by x, $\frac{1+2x}{2-5x} = \frac{\frac{1}{x}+2}{\frac{2}{x}-5}$.

The latter expression approaches the limit $\frac{0+2}{0-5}$ (§ 320), or $-\frac{2}{5}$, when x is indefinitely increased.

In any similar case, we divide both numerator and denominator of the fraction by the highest power of x.

EXERCISE 202

Find the limits approached by the following when x is indefinitely increased:

1. $\dfrac{4+5x-3x^2}{7-x+4x^2}$.
2. $\dfrac{2x+1}{3x^2-2}$.
3. $\dfrac{x^3-2x-4}{x^2+5x+3}$.

Find the values of the following:

4. $\dfrac{1}{x-2}-\dfrac{12}{x^3-8}$ when $x=2$.
5. $(2x^2-5x-3)\left(2+\dfrac{1}{x-3}\right)$ when $x=3$.

XXXIX. HORNER'S SYNTHETIC DIVISION

494. Division by Detached Coefficients.

In finding the quotient of two expressions which are arranged according to the same order of powers of some common letter, the operation may be abridged by writing only the *numerical coefficients* of the terms.

If the term involving any power is wanting, it may be supplied with the coefficient 0.

Ex. Divide $6x^6+2x^5-9x^4+5x^2+18x-30$ by $3x^3+x^2-6$.

$$\begin{array}{l|l}
6+2-9+\ 0+5+18-30 & 3+1+0-6 \\ \cline{2-2}
\underline{6+2+0-12} & 2+0-3+5 \\
\quad -9+12 & \\
\quad \underline{-9-\ 3+0+18} & \\
\qquad 15+5 & \\
\qquad \underline{15+5+\ 0-30} &
\end{array}$$

Then the quotient is $2x^3-3x+5$.

495. Horner's Synthetic Division.

Let it be required to divide $6x^4-x^3-3x^2+10x-12$ by $2x^2+x-3$.

$$\begin{array}{l|l}
6x^4-\ x^3-3x^2+10x-12 & 2x^2+x-3 \\ \cline{2-2}
\underline{6x^4+3x^3-9x^2} & 3x^2-2x+4 \\
\quad -4x^3+6x^2 & \\
\quad \underline{-4x^3-2x^2+\ 6x} & \\
\qquad 8x^2+\ 4x & \\
\qquad \underline{8x^2+\ 4x-12} &
\end{array}$$

The dividend equals $(2x^2+x-3)$ times the quotient.

Then, we can find the quotient by subtracting from the dividend $+x$ times the quotient, and -3 times the quotient, and dividing the result by $2x^2$.

Or, we can find it by *adding* to the dividend $-x$ times the quotient, and $+3$ times the quotient, and dividing the result by $2x^2$.

We may arrange the work as follows:

$$\begin{array}{r|rrrrr}
2x^2 & 6x^4 & -\ x^3 & -3x^2 & +10x & -12 \\
-x & & -3x^3 & +2x^2 & -\ 4x & \\
+3 & & & +9x^2 & -\ 6x & +12 \\
\hline
\multicolumn{4}{r}{3x^2-2x+4,\ \text{quo.}} & 0 & 0
\end{array}$$

We write the divisor in the left-hand column, with the sign of each term after the first changed.

We get the first term of the quotient, $3x^2$, by dividing the first term of the dividend, $6x^4$, by the first term of the divisor, $2x^2$.

Since we have to add to the dividend $-x$ times the quotient, and $+3$ times it, we can put down the terms $-3x^3$ in the second column, and $+9x^2$ in the third, these being the products of $-x$ and $+3$ by the first term of the quotient.

Now we get the second term of the quotient by subtracting $3x^3$ from $-x^3$, and dividing the result by $2x^2$.

Then, in the synthetic method, we add the terms in the second column, and divide the sum by $2x^2$, giving $-2x$.

We now write the term $+2x^2$ in the second column, and $-6x$ in the third; these being the products of $-x$ and $+3$ by the second term of the quotient.

We get the third term of the quotient by subtracting $-9x^2$ and $-2x^2$ from $-3x^2$, and dividing the result by $2x^2$.

Then, in the synthetic method, we add the terms in the third column, and divide the sum by $2x^2$, giving 4.

We now write the term $-4x$ in the fourth column, and $+12$ in the fifth; these being the products of $-x$ and $+3$ by the third term of the quotient.

We have now added to the dividend $-x$ times the quotient, and $+3$ times the quotient, and divided the result by $2x^2$; so that we have obtained the quotient.

The sum of the terms in the fourth and fifth columns is 0; if this had not been the case, there would have been a *remainder*.

496. We will now give some additional examples of the method:

1. Divide $\quad 12x^3 - 11x^2 + 20x - 9$ by $3x^2 - 2x + 4$.

$$
\begin{array}{r|l}
3x^2 & 12x^3 - 11x^2 + 20x - 9 \\
+2x & \qquad\quad + 8x^2 - \ \ 2x \\
-4 & \qquad\qquad\qquad - 16x + 4 \\
\hline
 & 4x \ - 1, \text{ quo.} \quad 2x - 5, \text{ Rem.}
\end{array}
$$

We write the divisor in the left-hand column, with the sign of each term after the first changed.

Dividing $12x^3$ by $3x^2$ gives $4x$ for the first term of the quotient.

We multiply $+2x$ by $4x$ and put the product, $8x^2$, in the second column; and multiply -4 by $4x$, and put the product, $-16x$, in the third column.

We add the terms in the second column, giving $-3x^2$, and divide the result by $3x^2$, giving -1 as the second term of the quotient.

We multiply $+2x$ by -1 and put the product, $-2x$, in the third column; and multiply -4 by -1, and put the product, $+4$, in the fourth column.

Adding the terms in the third and fourth columns, the sum is $2x - 5$

Then, the quotient is $4x - 1$, and the remainder $2x - 5$.

It is advantageous to use *detached coefficients* (§ 494) in the synthetic method; the work of Ex. 1 would then stand as follows:

$$
\begin{array}{r|rrrr}
3 & 12 & -11 & +20 & -9 \\
+2 & & +8 & -2 & \\
-4 & & & -16 & +4 \\
\hline
 & 4 & -1, & +2 & -5
\end{array}
$$

2. Divide

$$a^5 + 2a^4b - 14a^3b^2 + 15ab^4 - 5b^5 \text{ by } a^2 - 3ab + b^2.$$

$$
\begin{array}{r|llllll}
a^2 & a^5 & +2a^4b & -14a^3b^2 & & +15ab^4 & -5b^5 \\
+3ab & & +3a^4b & +15a^3b^2 & & -15ab^4 & \\
-b^2 & & & -\ \ a^3b^2 & -5a^2b^3 & & +5b^5 \\
\hline
 & a^3 & +5a^2b & +\ 0 & -5b^3, & \ \ 0 & \ \ 0
\end{array}
$$

In the above solution, the sum of the terms in the third column is 0, so that the third term of the quotient is 0.

We then add the terms in the fourth column and divide by a^2, giving $-5b^3$ as the fourth term of the quotient.

It is important, in an example like the above, to keep similar terms in the same vertical column,

The work of Ex. 2 will appear as follows with detached coefficients:

$$\begin{array}{r|rrrrrr}
1 & 1 & +2 & -14 & +0 & +15 & -5 \\
+3 & & +3 & +15 & & -15 & \\
-1 & & & -1 & -5 & & +5 \\
\hline
 & 1 & +5 & +0 & -5, & 0 & 0
\end{array}$$

EXERCISE 203

Divide the following by synthetic division:

1. $12x^3-7x^2-23x-3$ by $4x^2-5x-3$.
2. $4a^4-9a^2+30a-25$ by $2a^2+3a-5$.
3. $2a^4-a^3b+8ab^3-5b^4$ by $2a^2-3ab+5b^2$.
4. $4m^2n^4+n^8+16m^4$ by $2mn^2+4m^2+n^4$.
5. $6x^5-13x^4-20x^3+55x^2-14x-19$ by $2x^2-7x+6$.
6. $8x^5-4x^4y-8x^2y^3-18xy^4+21y^5$ by $4x^3-2x^2y+6xy^2-7y^3$.
7. $37a^2+50+a^5-70a$ by $2a^2+5+a^3-6a$.
8. $2a^2-ab-2ac-6b^2+11bc-4c^2$ by $2a+3b-4c$.

XL. PERMUTATIONS AND COMBINATIONS

497. The different orders in which things can be arranged are called their **Permutations.**

Thus, the permutations of the letters a, b, c, taken two at a time, are ab, ac, ba, bc, ca, cb; and their permutations, taken three at a time, are $abc, acb, bac, bca, cab, cba$.

498. The **Combinations** of things are the different collections which can be formed from them without regard to the order in which they are placed.

Thus, the combinations of the letters a, b, c, taken two at a time, are ab, bc, ca; for though ab and ba are different permutations, they form the same combination.

499. *To find the number of permutations of n different things taken two at a time.*

Consider the n letters $a, b, c, \cdots$.

In making any particular permutation of two letters, the first letter may be any one of the n; that is, the first place can be filled in n different ways.

After the first place has been filled, the second place can be filled with any one of the remaining $n-1$ letters.

Then, the whole number of permutations of the letters taken two at a time is $n(n-1)$.

We will now consider the general case.

500. *To find the number of permutations of n different things taken r at a time.*

Consider the n letters $a, b, c, \cdots$.

In making any particular permutation of r letters, the first letter may be any one of the n.

After the first place has been filled, the second place can be filled with any one of the remaining $n-1$ letters.

After the second place has been filled, the third place can be filled in $n-2$ different ways.

Continuing in this way, the rth place can be filled in

$$n-(r-1), \text{ or } n-r+1 \text{ different ways.}$$

Then, the whole number of permutations of the letters taken r at a time is given by the formula

$$_nP_r = n(n-1)(n-2)\cdots(n-r+1). \qquad (1)$$

The number of permutations of n different things taken r at a time is usually denoted by the symbol $_nP_r$.

501. If *all* the letters are taken, $r=n$, and (1) becomes

$$_nP_n = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1 = \underline{|n}. \qquad (2)$$

Hence, *the number of permutations of n different things taken n at a time equals the product of the natural numbers from 1 to n inclusive.* (See note, page 352.)

502. *To find the number of combinations of n different things taken r at a time.*

The number of *permutations* of n different things taken r at a time is $\quad n(n-1)(n-2)\cdots(n-r+1)$ (§ 500).

But, by § 501, each combination of r different things may have $\underline{|r}$ permutations.

Hence, the number of *combinations* of n different things taken r at a time equals the number of permutations divided by $\underline{|r}$.

That is, $$_nC_r = \frac{n(n-1)(n-2)\cdots(n-r+1)}{\underline{|r}}. \qquad (3)$$

The number of combinations of n different things taken r at a time is usually denoted by the symbol $_nC_r$.

503. Multiplying both terms of the fraction (3) by the product of the natural numbers from 1 to $n-r$ inclusive, we have

$$_nC_r = \frac{n(n-1)\cdots(n-r+1)\cdot(n-r)\cdots 2\cdot 1}{\underline{|r}\times 1\cdot 2\cdots(n-r)} = \frac{\underline{|n}}{\underline{|r}\,\underline{|n-r}};$$

which is another form of the result.

504. *The number of combinations of n different things taken r at a time equals the number of combinations taken $n-r$ at a time.*

For, for every selection of r things out of n, we leave a selection of $n-r$ things.

The theorem may also be proved by substituting $n-r$ for r, in the result of § 503.

505. Examples.

1. How many changes can be rung with 10 bells, taking 7 at a time?

Putting $n = 10$, $r = 7$, in (1), § 500,

$$_{10}P_7 = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 = 604800.$$

2. How many different combinations can be formed with 16 letters, taking 12 at a time?

By § 504, the number of combinations of 16 different things, taken 12 at a time, equals the number of combinations of 16 different things, taken 4 at a time.

Putting $n = 16$, $r = 4$, in (3), § 502,

$$_{16}C_4 = \frac{16 \cdot 15 \cdot 14 \cdot 13}{1 \cdot 2 \cdot 3 \cdot 4} = 1820.$$

3. How many different words, each consisting of 4 consonants and 2 vowels, can be formed from 8 consonants and 4 vowels?

The number of combinations of the 8 consonants, taken 4 at a time, is

$$\frac{8 \cdot 7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3 \cdot 4}, \text{ or } 70.$$

The number of combinations of the 4 vowels, taken 2 at a time, is

$$\frac{4 \cdot 3}{1 \cdot 2}, \text{ or } 6.$$

Any one of the 70 sets of consonants may be associated with any one of the 6 sets of vowels; hence, there are in all 70×6, or 420 sets, each containing 4 consonants and 2 vowels.

But each set of 6 letters may have $\underline{|6}$, or 720 different permutations (§ 501).

Therefore, the whole number of different words is

$$420 \times 720, \text{ or } 302400.$$

EXERCISE 204

1. How many different permutations can be formed with 14 letters, taken 6 at a time?

2. In how many different orders can the letters in the word *triangle* be written, taken all together?

3. How many combinations can be formed with 15 things, taken 5 at a time?

4. A certain play has 5 parts, to be taken by a company of 12 persons. In how many different ways can they be assigned?

5. How many combinations can be formed with 17 things, taken 11 at a time?

6. How many different numbers, of 6 different figures each, can be formed from the digits 1, 2, 3, 4, 5, 6, 7, 8, 9, if each number begins with 1, and ends with 9?

7. How many even numbers, of 5 different figures each, can be formed from the digits 4, 5, 6, 7, 8?

8. How many different words, of 8 different letters each, can be formed from the letters in the word *ploughed*, if the third letter is *o*, the fourth *u*, and the seventh *e*?

9. How many different committees, of 8 persons each, can be formed from a corporation of 14 persons? In how many will any particular individual be found?

10. There are 11 points in a plane, no 3 in the same straight line. How many different quadrilaterals can be formed, having 4 of the points for vertices?

11. From a pack of 52 cards, how many different hands of 6 cards each can be dealt?

12. A and B are in a company of 48 men. If the company is divided into equal squads of 6, in how many of them will A and B be in the same squad?

13. How many different words, each having 5 consonants and 1 vowel, can be formed from 13 consonants and 4 vowels?

14. Out of 10 soldiers and 15 sailors, how many different parties can be formed, each consisting of 3 soldiers and 3 sailors?

15. A man has 22 friends, of whom 14 are males. In how many ways can he invite 16 guests from them, so that 10 may be males?

16. From 3 sergeants, 8 corporals, and 16 privates, how many different parties can be formed, each consisting of 1 sergeant, 2 corporals, and 5 privates?

17. Out of 3 capitals, 6 consonants, and 4 vowels, how many different words of 6 letters each can be formed, each beginning with a capital, and having 3 consonants and 2 vowels?

18. How many different words of 8 letters each can be formed from 8 letters, if 4 of the letters cannot be separated? How many if these 4 can only be in one order?

19. How many different numbers, of 7 figures each, can be formed from the digits 1, 2, 3, 4, 5, 6, 7, 8, 9, if the first, fourth, and last digits are odd numbers?

506. *To find the number of permutations of n things which are not all different, taken all together.*

Let there be n letters, of which p are a's, q are b's, and r are c's, the rest being all different.

Let N denote the number of permutations of these letters taken all together.

Suppose that, in any particular permutation of the n letters, the p a's were replaced by p new letters, differing from each other and also from the remaining letters.

Then, by simply altering the order of these p letters among themselves, without changing the positions of any of the other letters, we could from the original permutation form $\lfloor p$ different permutations (§ 501).

If this were done in the case of each of the N original permutations, the whole number of permutations would be $N \times \lfloor p$.

Again, if in any one of the latter the q b's were replaced by q new letters, differing from each other and from the remaining letters, then by altering the order of these q letters among themselves, we could from the original permutation form $\underline{|q}$ different permutations; and if this were done in the case of each of the $N\times\underline{|p}$ permutations, the whole number of permutations would be $N\times\underline{|p}\times\underline{|q}$.

In like manner, if in each of the latter the r c's were replaced by r new letters, differing from each other and from the remaining letters, and these r letters were permuted among themselves, the whole number of permutations would be

$$N\times\underline{|p}\times\underline{|q}\times\underline{|r}.$$

We now have the original n letters replaced by n *different* letters.

But the number of permutations of n different things taken n at a time is $\underline{|n}$ (§ 501).

Therefore, $N\times\underline{|p}\times\underline{|q}\times\underline{|r}=\underline{|n}$; or, $N=\dfrac{\underline{|n}}{\underline{|p}\,\underline{|q}\,\underline{|r}}$.

Any other case can be treated in a similar manner.

Ex. How many permutations can be formed from the letters in the word *Tennessee*, taken all together?

Here there are 4 e's, 2 n's, 2 s's, and 1 t.
Putting in the above formula $n=9$, $p=4$, $q=2$, $r=2$, we have

$$\frac{\underline{|9}}{\underline{|4}\,\underline{|2}\,\underline{|2}}=\frac{5\cdot 6\cdot 7\cdot 8\cdot 9}{2\cdot 2}=3780.$$

EXERCISE 205

1. In how many different orders can the letters of the word *denomination* be written?

2. There are 4 white billiard balls exactly alike, and 3 red balls, also alike; in how many different orders can they be arranged?

3. In how many different orders can the letters of the word *independence* be written?

4. How many different signals can be made with 7 flags, of which 2 are blue, 3 red, and 2 white, if all are hoisted for each signal?

5. How many different numbers of 8 digits can be formed from the digits 4, 4, 3, 3, 3, 2, 2, 1?

6. In how many different ways can 2 dimes, 3 quarters, 4 halves, and 5 dollars be distributed among 14 persons, so that each may receive a coin?

507. *To find for what value of r the number of combinations of n different things taken r at a time is greatest.*

By § 502, the number of combinations of n different things, taken r at a time, is

$$ {}_nC_r = \frac{n(n-1)\cdots(n-r+2)(n-r+1)}{1\cdot 2\cdot 3\cdots(r-1)r}. \quad (1)$$

Also, the number of combinations of n different things, taken $r-1$ at a time, is

$$\frac{n(n-1)\cdots[n-(r-1)+1]}{1\cdot 2\cdot 3\cdots(r-1)}, \text{ or } \frac{n(n-1)\cdots(n-r+2)}{1\cdot 2\cdot 3\cdots(r-1)}. \quad (2)$$

The expression (1) is obtained by multiplying the expression (2) by $\frac{n-r+1}{r}$, or $\frac{n+1}{r}-1$.

The latter expression decreases as r increases.

If, then, we find the values of (1) corresponding to the values 1, 2, 3, ⋯, of r, the results will continually increase so long as $\frac{n-r+1}{r}$ is >1.

I. Suppose n even; and let $n=2m$, where m is a positive integer.

Then, $\frac{n-r+1}{r}$ becomes $\frac{2m-r+1}{r}$.

If $r=m$, $\frac{2m-r+1}{r}$ becomes $\frac{m+1}{m}$, and is >1.

If $r=m+1$, $\frac{2m-r+1}{r}$ becomes $\frac{m}{m+1}$, and is <1.

Then, ${}_nC_r$ will have its greatest value when $r=m=\frac{n}{2}$.

II. Suppose n odd; and let $n=2m+1$, where m is a positive integer.

Then, $\frac{n-r+1}{r}$ becomes $\frac{2m-r+2}{r}$.

If $r=m$, $\frac{2m-r+2}{r}$ becomes $\frac{m+2}{m}$, and is >1.

If $r=m+1$, $\frac{2m-r+2}{r}$ becomes $\frac{m+1}{m+1}$, and equals 1.

If $r=m+2$, $\frac{2m-r+2}{r}$ becomes $\frac{m}{m+2}$, and is <1.

Then, ${}_nC_r$ will have its greatest value when r equals m or $m+1$; that is, $\frac{n-1}{2}$ or $\frac{n-1}{2}+1$.

Then, ${}_nC_r$ will have its greatest value when r equals $\frac{n-1}{2}$ or $\frac{n+1}{2}$; the results being the same in these two cases.

XLI. EXPONENTIAL AND LOGARITHMIC SERIES

508. The Theorem of Limits.

If two expressions, containing the same variable (§ 317), *are equal for every value of the variable, and each approaches a limit* (§ 318), *the limits are equal.*

Let A and B be two expressions containing the same variable.

Let A and B be equal for every value of the variable, and approach the limits A' and B', respectively.

To prove $$A' = B'.$$

Let $$A' - A = m, \text{ and } B' - B = n.$$

Then, m and n are variables which can be made less than any assigned fixed number, however small (§ 318).

Then, either $m - n$ is a variable which can be made less than any assigned fixed number, however small, or else $m - n = 0$.

But
$$m - n = A' - A - (B' - B)$$
$$= A' - A - B' + B = A' - B';$$

for, by hypothesis, A and B are equal for every value of the variable.

But $A' - B'$ is not a variable; and hence $m - n$ is not a variable.

Then, $m - n$ is 0; and hence its equal, $A' - B'$, is 0, or

$$A' = B'.$$

THE EXPONENTIAL SERIES

509. We have for all values of n and x,

$$\left[\left(1+\frac{1}{n}\right)^n\right]^x = \left(1+\frac{1}{n}\right)^{nx}.$$

Expanding both members by the Binomial Theorem,

$$\left[1+n\cdot\frac{1}{n}+\frac{n(n-1)}{\underline{|2}}\cdot\frac{1}{n^2}+\frac{n(n-1)(n-2)}{\underline{|3}}\cdot\frac{1}{n^3}+\cdots\right]^x$$

$$=1+nx\cdot\frac{1}{n}+\frac{nx(nx-1)}{\underline{|2}}\cdot\frac{1}{n^2}$$

$$+\frac{nx(nx-1)(nx-2)}{\underline{|3}}\cdot\frac{1}{n^3}+\cdots. \qquad (1)$$

We may write equation (1) in the form

$$\left[1+1+\frac{1-\frac{1}{n}}{\underline{|2}}+\frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{\underline{|3}}+\cdots\right]^x$$

$$=1+x+\frac{x\left(x-\frac{1}{n}\right)}{\underline{|2}}+\frac{x\left(x-\frac{1}{n}\right)\left(x-\frac{2}{n}\right)}{\underline{|3}}+\cdots; \qquad (2)$$

which holds however great n may be.

Now let n be indefinitely increased.

Then, the limit of each of the terms $\frac{1}{n}$, $\frac{2}{n}$, etc., is 0 (§ 320).

Hence, the limiting value of the first member of (2) is

$$\left[1+1+\frac{1}{\underline{|2}}+\frac{1}{\underline{|3}}+\cdots\right]^x,$$

and the limiting value of the second member is

$$1+x+\frac{x^2}{\underline{|2}}+\frac{x^3}{\underline{|3}}+\cdots.$$

By the Theorem of Limits (§ 508), these limits are equal; that is,

$$\left[1+1+\frac{1}{\underline{|2}}+\frac{1}{\underline{|3}}+\cdots\right]^x=1+x+\frac{x^2}{\underline{|2}}+\frac{x^3}{\underline{|3}}+\cdots.$$

Denoting the series in brackets by e, we obtain

$$e^x=1+x+\frac{x^2}{\underline{|2}}+\frac{x^3}{\underline{|3}}+\cdots. \qquad (3)$$

510. Putting mx for x, in (3), § 509, we have

$$e^{mx} = 1 + mx + \frac{m^2x^2}{\underline{|2}} + \frac{m^3x^3}{\underline{|3}} + \cdots. \qquad (4)$$

Let $m = \log_e a$.

Then, by § 412, $e^m = a$, and $e^{mx} = a^x$.

Substituting in (4), we obtain

$$a^x = 1 + (\log_e a)x + (\log_e a)^2\frac{x^2}{\underline{|2}} + (\log_e a)^3\frac{x^3}{\underline{|3}} + \cdots. \qquad (5)$$

This result is called the *Exponential Series.*

511. The system of logarithms which has e for its base is called the *Napierian System,* from Napier, the inventor of logarithms.

Napierian logarithms are also called *Natural Logarithms.*

The approximate value of e may be readily calculated from the series of § 509.

$$e = 1 + 1 + \frac{1}{\underline{|2}} + \frac{1}{\underline{|3}} + \cdots,$$

and will be found to equal **2.7182818** ….

THE LOGARITHMIC SERIES

512. *To expand* $\log_e(1+x)$ *in ascending powers of* x.

Substituting in (5), § 510, $1 + x$ for a, and y for x,

$$(1+x)^y = 1 + [\log_e(1+x)]y + \text{terms in } y^2, y^3, \text{etc.}$$

Expanding the first member by the Binomial Theorem,

$$1 + yx + \frac{y(y-1)}{\underline{|2}}x^2 + \frac{y(y-1)(y-2)}{\underline{|3}}x^3 + \cdots$$

$$= 1 + [\log_e(1+x)]y + \text{terms in } y^2, y^3, \text{etc.} \qquad (6)$$

This equation holds for every value of y which makes both members convergent; and, by the Theorem of Undetermined Coefficients (§ 396), the coefficients of y in the two series are equal.

That is, $x - \frac{\lfloor 1}{\lfloor 2} x^2 + \frac{\lfloor 2}{\lfloor 3} x^3 - \frac{\lfloor 3}{\lfloor 4} x^4 + \cdots = \log_e (1+x).$

Or, $$\log_e (1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots. \qquad (7)$$

This result is called the **Logarithmic Series.**

CALCULATION OF LOGARITHMS

513. The equation (7), § 512, can be used to calculate Napierian Logarithms, if x is so taken that the second member is convergent; but unless x is small, it requires the sum of a great many terms to insure any degree of accuracy.

We will now derive a more convenient series for the calculation of Napierian Logarithms.

514. Putting $-x$ for x, in (7), § 512, we have

$$\log_e (1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \cdots. \qquad (8)$$

Subtracting (8) from (7), we obtain

$$\log_e (1+x) - \log_e (1-x) = 2x + \frac{2x^3}{3} + \frac{2x^5}{5} + \cdots.$$

Or (§ 422), $$\log_e \frac{1+x}{1-x} = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right). \qquad (9)$$

Let $x = \frac{m-n}{m+n}$; then $\frac{1+x}{1-x} = \frac{1 + \frac{m-n}{m+n}}{1 - \frac{m-n}{m+n}} = \frac{2m}{2n} = \frac{m}{n}.$

Substituting these values in (9), we obtain

$$\log_e \frac{m}{n} = 2\left[\frac{m-n}{m+n} + \frac{1}{3}\left(\frac{m-n}{m+n}\right)^3 + \frac{1}{5}\left(\frac{m-n}{m+n}\right)^5 + \cdots\right].$$

But by § 422, $\log_e \frac{m}{n} = \log_e m - \log_e n$; whence,

$$\log_e m = \log_e n + 2\left[\frac{m-n}{m+n} + \frac{1}{3}\left(\frac{m-n}{m+n}\right)^3 + \frac{1}{5}\left(\frac{m-n}{m+n}\right)^5 + \cdots\right].$$

515. Let it be required, for example, to calculate the Napierian logarithm of 2 to six places of decimals.

Putting $m=2$ and $n=1$ in the result of § 514, we have

$$\log_e 2 = \log_e 1 + 2\left[\frac{1}{3}+\frac{1}{3}\left(\frac{1}{3}\right)^3+\frac{1}{5}\left(\frac{1}{3}\right)^5+\cdots\right].$$

Or since $\log_e 1 = 0$ (§ 418),

$$\begin{aligned}\log_e 2 &= 2(.3333333+.0123457+.0008230+.0000653\\ &\qquad +.0000056+.0000005+\cdots)\\ &= 2\times .3465734 = .6931468 = .693147,\end{aligned}$$

correct to six places of decimals.

Having found $\log_e 2$, we may calculate $\log_e 3$ by putting $m=3$ and $n=2$ in the result of § 514.

Proceeding in this way, we shall find $\log_e 10 = 2.302585\cdots$.

516. *To calculate the common logarithm of a number, having given its Napierian logarithm.*

Putting $b=10$ and $a=e$ in the result of § 426,

$$\log_{10} m = \frac{\log_e m}{\log_e 10} = \frac{1}{2.302585}\times \log_e m = .4342945\times \log_e m.$$

Thus, $\qquad \log_{10} 2 = .4342945 \times .693147 = .301030.$

The multiplier by which logarithms of any system are derived from Napierian logarithms is called the *modulus* of that system.

Thus, .4342945 is the modulus of the common system.

Conversely, to find the Napierian logarithm of a number when its common logarithm is given, we may either divide the common logarithm by the modulus .4342945, or multiply it by 2.302585, the reciprocal of .4342945.

EXERCISE 206

Using the table of common logarithms, find the Napierian logarithm of each of the following to four significant figures:

1. 10000. **2.** .001. **3.** 9.93.

4. 243.6. 5. .04568. 6. .56734.

7. What is the characteristic of $\log_5 758$?

8. What is the characteristic of $\log_7 500$?

9. If $\log 3 = .4771$, how many digits are there in 3^{17}?

10. If $\log 8 = .9031$, how many digits are there in 8^{23}?

THE METRIC SYSTEM

Linear Measure

The standard unit of Linear Measure in the Metric System is the **Meter**. It is determined by taking one ten-millionth part of the distance from the earth's equator to either of its poles, measured on a meridian. It is equal to 39.37 inches.

The problems in this book make use of the following subdivisions of the Meter:

10 Millimeters (mm.)	=	1 Centimeter (cm.)
10 Centimeters	=	1 Decimeter (dm.)
10 Decimeters	=	1 Meter (m.)

Measures of Weight

The **Gram** is the unit of weight. It is equal to the weight of a cubic centimeter of distilled water at its greatest density.

The following multiples of the gram are used in problems in this book:

10 Grams (g.)	=	1 Dekagram (Dg.)
10 Dekagrams	=	1 Hektogram (Hg.)
10 Hektograms	=	1 Kilogram (Kg.)

INDEX

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Common monomial factor

2 terms

Difference of 2 perfect squares

$$a^2 - b^2 = (a+b)(a-b)$$

Sum of 2 cubes

Difference of 2 cubes

Sum or difference of 2 odd power

3 terms

In form of a perfect square

$$a^2 + 2ab + b^2 = (a+b)^2$$

not in the form of a perfect square

$$a^2 + 5a + 6 = (a+3)(a+2)$$

Reduce all terms to the form of perfect square.

4 terms

Monomial factor common to first 2 terms & then last two

$1^2 = 1$

$2^2 = 4$

$3^2 = 9$

$4^2 = 16$

$5^2 = 25$

$6^2 = 36$

$7^2 = 49$

$8^2 = 64$

$9^2 = 81$

$10^2 = 100$

$11^2 = 121$

$12^2 = 144$

$13^2 = 1$[illegible]

$14^2 = 196$

$15^2 = 225$

[illegible] $= 256$

[illegible]$^2 = 4$[illegible]

[illegible]$^2 = 6$[illegible]

$1^3 = 1$

$2^3 = 8$

$3^3 = 27$

$4^3 = 64$

$5^3 = 125$

$6^3 = 216$

$7^3 = 543$

$8^3 = 512$

$9^3 = 729$

$10^3 = 1000$